Additive-Belief-Based Preferences*

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Abstract

We introduce a new class of preferences — which we call additive-belief-based (ABB) utility — that captures a general, but still tractable, approach to belief-based utility, and that encompasses many popular models in the behavioral literature. We axiomatize a general class of ABB preferences, as well as two prominent special cases that allow utility to depend on the level of each period's beliefs but not on changes in beliefs across periods. We also identify the intersection of ABB preferences with the class of recursive preferences and characterize attitudes towards the timing of resolution of uncertainty for such preferences.

JEL codes: D80, D81

Key words: Anticipatory utility, Compound lotteries, Preferences over beliefs, Recursive preferences, Resolution of uncertainty.

1 Motivation

It is both intuitive and well documented that beliefs about future consumption or life events are likely to directly affect well-being. For example, an individual may enjoy looking forward to an upcoming vacation and particularly so if the risk of severe weather conditions became very unlikely; on the other hand, the same individual may worry about a future medical procedure he determined to undertake. Loewenstein (1987) used a survey technique to demonstrate the effect of anticipation motives. There is also evidence based on fMRI studies that experience and anticipation of pain produce psychological-stress reactions (Berns et. al., 2006; Lazarus, 1966).

Consequently, models of decision-making in which individuals derive utility not only from material outcomes but also from their current and future beliefs have become increasingly prominent in economics (see, for example, Caplin and Leahy, 2001; Kőszegi and Rabin, 2009; and Ely, Frankel,

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and Kamenica, 2015). In most of these models, overall utility is additively separable between material payoffs and beliefs. Our aim in this paper is to introduce and analyze a new class of utility functions — which we call *additive-belief-based (ABB)* utility — that captures a general additively separable approach to belief-based utility.

Models within this class usually take one of two forms. The first, as in Caplin and Leahy (2001), allows individuals' utility to depend on the (absolute) level of beliefs; i.e., on how likely it is that certain states/payoffs occur. In line with previous literature, we refer to these set of models as *anticipatory utility* models. In the second, as in Kőszegi and Rabin (2009), utility depends not on the level of beliefs, but on changes in beliefs in any given period. We refer to this class as *changing beliefs* models. Our ABB representation encompasses both these frameworks. Moreover, we point out a useful partition of the class of anticipatory utility models into (i) *prior-anticipatory utility* models, where utility depends on beliefs at the beginning of the time period, before information has been received; and (ii) *posterior-anticipatory utility* models, where utility depends on beliefs at the end of the time period, after information has been received.

While individuals in these models gain utility from their beliefs, they cannot directly choose them. Rather, individuals begin with a prior belief, receive information, and form interim beliefs by applying Bayes' rule. Therefore, individuals can control their beliefs only by choosing particular information structures. And since individuals gain utility from their beliefs, they may exhibit preferences over information structures even if they cannot react to new information by altering their behavior, that is, even if they do not have actions to take in the interim stage. This feature is the one that distinguishes these models from the standard model, in which individuals would be indifferent between all possible information structures when no actions are available. To tightly link these models to observable behavior, we look at preferences over the combination of information structures and prior beliefs. These can be elicited in a natural way in experimental and field settings. Formally, taking advantage of the natural mapping between information structures and compound lotteries, we take as our domain of preferences the set of two-stage compound lotteries, that is, lotteries whose prizes are different lotteries over final outcomes.

In order to provide some intuition regarding the class of ABB functions, let P be a typical two-stage compound lottery. In period 0 it induces a prior belief $\phi(P)$; the individual knows that the overall probability to receive x_j in period 2 is $\phi(P)(x_j)$. In period 1, P generates a signal i with probability $P(p_i)$. Signal i generates a posterior belief over outcomes; the individual now knows that in period 2 they will receive x_j with probability $p_i(x_j)$. In period 2, all uncertainty resolves and the individual receives x_j and has degenerate beliefs centering on this outcome (denoted δ_{x_j}). The total utility of this scenario, denoted $V_{ABB}(P)$, is given by:

$$
\begin{aligned}
V_{ABB}(P) \;=\; & \underbrace{\sum_j \phi(P)(x_j)u(x_j)}_{\text{expected utility from material payoffs}} \\
+\; & \underbrace{\sum_i P(p_i)\nu_1(\phi(P), p_i)}_{\text{expected utility from beliefs in periods 0 and 1}} \\
+\; & \underbrace{\sum_i P(p_i)\sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})}_{\text{expected utility from beliefs in periods 1 and 2}}
\end{aligned}
$$

The first term represents the expected consumption utility of the two-stage lottery — the expected utility that the individual receives from material outcomes (in period 2).[1] The second term represents period 1's belief-based utility — the individual's expected utility from having interim beliefs p_i in period 1, conditional on having prior beliefs $\phi(P)$. The last term represents period 2's belief-based utility — the individual's expected utility from x_j being realized in period 2, conditional on having interim beliefs p_i.

As a concrete example, suppose there are two outcomes: H (high) and L (low), so that beliefs are summarized by the probability of H. Suppose the prior beliefs is that the two outcomes are equally likely. Then the expected Bernoulli utility over material outcomes is $\frac{1}{2}u(H) + \frac{1}{2}u(L)$. In Period 1, the individual gets a binary signal: half the time its good, and beliefs move to $\frac{3}{4}$ with corresponding utility $\nu_1(\frac{1}{2}, \frac{3}{4})$; half the time its bad, and beliefs fall to $\frac{1}{4}$ with corresponding utility $\nu_1(\frac{1}{2}, \frac{1}{4})$. Expected belief-based utility in Period 1 is thus

$$\frac{1}{2}\nu_1(\frac{1}{2}, \frac{3}{4}) + \frac{1}{2}\nu_1(\frac{1}{2}, \frac{1}{4}).$$

In Period 2, the individual learns for sure whether he got H or L. After the good signal utility is either $\nu_2(\frac{3}{4}, 1)$ or $\nu_2(\frac{3}{4}, 0)$; and after the bad signal utility is either $\nu_2(\frac{1}{4}, 1)$ or $\nu_2(\frac{1}{4}, 0)$. Expected belief based utility in Period 2 is thus

$$\frac{1}{2}\left(\frac{3}{4}\nu_2(\frac{3}{4}, 1) + \frac{1}{4}\nu_2(\frac{3}{4}, 0)\right) + \frac{1}{2}\left(\frac{1}{4}\nu_2(\frac{1}{4}, 1) + \frac{3}{4}\nu_2(\frac{1}{4}, 0)\right).$$

Overall ABB utility is the sum of these three components.

We first show how the prominent sub-classes of ABB functionals are related to each other, demonstrating that models that allow for changing beliefs nest those of prior anticipatory beliefs, which in turn nest those of posterior anticipatory beliefs. We then provide necessary and sufficient

[1] Note that $\phi(P)(x_j) = \sum_i P(p_i) \sum_j p_i(x_j)$.

conditions for continuous preferences to be represented with an ABB functional. We show that ABB utility is characterized by a single relaxation of the standard vNM-Independence axiom (applied to preferences over compound lotteries): The Conditional Two-Stage Independence (CTI) axiom requires Independence (in mixing the compound lotteries) to hold only if *all* compound lotteries involved in the mixing induce the same prior distribution over final outcomes. That is, if $\phi(P) = \phi(Q) = \phi(R)$, then P is preferred to Q if and only if the mixture of P and R is preferred to the (same-proportion) mixture of Q and R.

We next turn to showing how strengthening CTI allows us to characterize models of prior-anticipatory beliefs. The key behavior that distinguishes utility from changes in beliefs and utility from the level of beliefs is how broadly Independence (again over compound lotteries) holds. If individuals only care about the levels of their beliefs, then Independence should hold whenever the two lotteries involved in the initial comparison, but not necessarily the one they are both mixed with, induce the same prior distribution over outcomes. That is, the Strong Conditional Two-Stage Independence (SCTI) axiom drops from the previous axiom the requirement that $\phi(R)$ agrees with $\phi(P) = \phi(Q)$. We last show that adding a version of one-stage Independence — that is, Independence imposed on a specific subset of degenerate two-stage lotteries — to the previous axioms characterizes posterior-anticipatory beliefs. Thus, our results demonstrate how a simple set of familiar and easily tested conditions in terms of observed behavior allows distinguishing between different types of belief-dependent utility.

ABB preferences are not the only class of preference that have been developed to explain informational preferences, even in the absence of the ability to condition actions on that information. A different vein of the literature, primarily developed by Kreps and Porteus (1978) and extended by Segal (1990), focuses on recursive preferences over compound lotteries (and information). We show that in the context of two-stage compound lotteries, the intersection of the two models is precisely the class of preferences that admit a posterior-anticipatory beliefs representation.

Lastly, we analyze what types of restrictions on the functional forms of ABB preferences are equivalent to well-known types of intrinsic (i.e., non-instrumental) informational preferences, such as preferences for early resolution of uncertainty (Kreps and Porteus, 1978) or preferences for one-shot resolution of uncertainty (Dillenberger, 2010). In doing so, we provide characterizations that generalize some earlier results, for example those of Kőszegi and Rabin (2009), which were made in the context of specific functional forms.[2] Our results will allow us to compare how different classes of models (recursive and ABB) can, or cannot, accommodate different non-instrumental attitudes towards information.

[2] The specification of Kőszegi and Rabin (2009) functional form is as follows: given lottery p, let $c_p(\iota)$ be the payoff at percentile ι of the distribution induced by p. Then $\nu_1 = \kappa_1 \int \mu(u(c_{\phi(P)}(\iota) - u(c_{p_i}(\iota)))d\iota$, $\nu_2 = \kappa_2 \int \mu(u(c_{p_i}(\iota)) - u(c_{\delta_x}(\iota)))d\iota$, where μ is a gain-loss utility function that is continuous, strictly increasing, twice differentiable for $x \neq 0$ with $\mu''(x) \leq 0$ for $x > 0$ and $\mu''(x) \geq 0$ for $x < 0$, and satisfying $\mu(0) = 0$, $\mu(y) + \mu(-y) < \mu(x) + \mu(-x)$ whenever $y < x \geq 0$, and $\frac{\lim_{x\to 0} \mu''(|x|)}{\lim_{x\to 0} \mu''(-|x|)} = \lambda > 1$.

2 Model, Characterization, and Special Cases

Preliminaries:

Consider a set of prizes X, which is assumed to be a closed subset of some metric space. A simple lottery p on X is a probability distribution over X with a finite support. Let $\Delta(X)$ (or simply Δ) be the set of all simple lotteries on X. For any lotteries $p, q \in \Delta$ and $\alpha \in (0,1)$, we let $\alpha p + (1-\alpha)q$ be the lottery that yields prize x with probability $\alpha p(x) + (1-\alpha)q(x)$. Denote by δ_x the degenerate lottery that yields x with probability 1 and let $\bar{X} = \{\delta_x : x \in X\}$; we will often abuse notation and refer to δ_x simply as x. Similarly, denote by $\Delta(\Delta(X))$ (or simply Δ^2) the set of simple lotteries over Δ, that is, compound lotteries. For $P, Q \in \Delta^2$ and $\alpha \in (0,1)$, denote by $R = \alpha P + (1-\alpha)Q$ the lottery that yields simple (one-stage) lottery p with probability $\alpha P(p) + (1-\alpha)Q(p)$. Denote by D_p the degenerate, in the first stage, compound lottery that yields p with certainty. We sometime write a lottery (over either Δ or Δ^2) explicitly as a list; for example, $P = (p_1, P(p_1); \ldots; p_n; P(p_n))$, or $((p_i, P(p_i))_{i=1}^n$, denotes the two-stage lottery that, for $i = 1, ..., n$, yields p_i with probability $P(p_i)$. Define a *reduction operator* $\phi : \Delta^2 \to \Delta$ that maps compound lotteries to reduced one-stage lotteries by $\phi(Q) = \sum_{p \in \Delta} Q(p)p$.[3]

Two special subsets of compound lotteries are (i) $\Gamma = \{D_p | p \in \Delta\}$, the set of degenerate lotteries in Δ^2. Γ is the set of late resolving lotteries, where any $P \in \Gamma$ captures a situation in which the information structure reveals no information in period 1, so that the interim posterior and prior beliefs coincide; and (ii) $\Lambda = \{Q \in \Delta^2 | Q(p) > 0 \Rightarrow p \in \bar{X}\}$, the set of compound lotteries whose outcomes are degenerate in Δ. Λ is the set of early resolving lotteries, where any $P \in \Lambda$ captures a situation in which the information structure reveals all information in period 1, so that interim posteriors after observing any signal have one element in their support.

Our primitive is a binary relation $\succsim$ over Δ^2. We define the restriction of $\succsim$ to the subsets Γ and Λ as $\succsim_\Gamma$ and $\succsim_\Lambda$, respectively.[4]

Functional Forms:

We first formally define additive-belief-based utility.

Definition 1. *An additive-belief-based (ABB) representation is a tuple (u, ν_1, ν_2) consisting of continuous functions $u : X \to \mathbb{R}$, $\nu_1 : \Delta \times \Delta \to \mathbb{R}$, and $\nu_2 : \Delta \times \bar{X} \to \mathbb{R}$, such that $V_{ABB} : \Delta^2 \to \mathbb{R}$ defined as*

[3] Such compound lotteries are isomorphic to the set of prior beliefs over outcomes X, plus a potential information structure. We can simply associate an information structure with the set of posterior beliefs it induces — the set of second-stage lotteries.

[4] Note that both Γ and Λ are isomorphic to Δ, and therefore $\succsim_\Gamma$ and $\succsim_\Lambda$ can be interpreted as the the individual's preferences over simple lotteries in the appropriate period.

$$V_{ABB}(P) = \sum_j \phi(P)(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i) + \sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$$

represents $\succsim$.

The general ABB functional form allows utility to depend on changes in beliefs in period 1 and period 2. If utility depends on changes in beliefs, then ν_1 and ν_2 are functions of both their arguments. Alternatively, many models in the literature assume that individuals do not care about the changes in their beliefs, but rather care about the levels of their beliefs. Individuals may care about their beliefs in any given period in one of two ways. The first case supposes that utility depends on beliefs at the beginning of any period, that is, ν_1 is solely a function of $\phi(P)$ and ν_2 is solely a function of p_i. We call this functional form prior-anticipatory utility and define it as follows.

Definition 2. *A prior-anticipatory representation is an ABB representation with the restrictions that* $\nu_1(\phi(P), p_i) = \hat{\nu_1}(\phi(P))$ *and* $\nu_2(p_i, \delta_{x_j}) = \hat{\nu_2}(p_i)$.

In the second case, utility is derived from beliefs at the end of any period (that period's posterior beliefs, after receiving information), that is, ν_1 is solely a function of p_i and ν_2 is solely a function of δ_{x_j}. We call this posterior-anticipatory utility and the functional form is given by:

Definition 3. *A posterior-anticipatory representation is an ABB representation with the restrictions that* $\nu_1(\phi(P), p_i) = \bar{\nu_1}(p_i)$ *and* $\nu_2(p_i, \delta_{x_j}) = \bar{\nu_2}(\delta_{x_j})$.

Clearly, both anticipatory representations above are subsets of V_{ABB}. More surprisingly, prior-anticipatory representation nests posterior-anticipatory representation.

Lemma 1. *If* $\succsim$ *has a posterior-anticipatory representation, then it has a prior-anticipatory representation.*

Characterization:

We now characterize the functionals we have described using the relation $\succsim$. As will become apparent, our approach to restrict preferences is to impose Independence-type conditions on particular subsets of Δ^2. The first two axioms are standard.

Weak Order (WO) *The relation* $\succsim$ *is complete and transitive.*

Continuity (C) *The relation* $\succsim$ *is continuous.*

Our key axiom is Conditional Two-Stage Independence (CTI). CTI requires the Independence axiom to hold within the set of compound lotteries which share the same reduced form probabilities over outcomes (the same ϕ). Observe that the set $\{Q \in \Delta^2 | \phi(Q) = p\}$ is convex for any $p \in \Delta$. Thus, CTI says that Independence holds along "slices" of the compound lottery space, where all elements of the slice have the same reduced form probabilities.

Conditional Two-Stage Independence (CTI): *Suppose* $\phi(P) = \phi(P') = \phi(Q)$. *Then* $P \succsim P'$ *if and only if* $\alpha P + (1-\alpha)Q \succsim \alpha P' + (1-\alpha)Q$.

Axiom CTI can be further motivated through the lens of preferences for information. Recall that we identify preferences over compound lotteries with preferences over the combination of information structures and prior beliefs. CTI then requires that within a set of information structures that correspond to the same prior beliefs, the individual is an expected utility maximizer; "non-standard" behavior may arise only when we change the underlying prior beliefs.

Our first main result shows that CTI, along with the standard two axioms above, is all we need to characterize preferences that admit an ABB representation.

Proposition 1. *The relation* $\succsim$ *satisfies WO, C, and CTI, if and only if it has an ABB representation.*

CTI is not so restrictive, as it requires mixing not to reverse rankings only when all lotteries involved in the mixing have the same reduced form probabilities. A natural way to strengthen CTI is to suppose that only the compound lotteries involved in the original preference comparison need to have the same reduced form probabilities; the common compound lottery that they are mixed with need not. This means that the pair of lotteries which are compared after the mixing will have the same reduced form probabilities as each other, but need not have the same reduced form probabilities as the original pair. We call the axiom which formalizes this intuition Strong Conditional Two-Stage Independence (SCTI).

Strong Conditional Two-Stage Independence (SCTI): *Suppose* $\phi(P) = \phi(P')$. *Then* $P \succsim P'$ *if and only if* $\alpha P + (1-\alpha)Q \succsim \alpha P' + (1-\alpha)Q$.

SCTI rules out complementarity between the prior distribution and the corresponding information systems. That is, irrespectively of the underlying prior beliefs, the individual consistently chooses among information systems based on the expected utility criterion; violations of expected utility may occur only when ranking compound lotteries that do not refine the same prior beliefs.

SCTI clearly implies CTI. It also gives us our second characterization result:

Proposition 2. *The relation* $\succsim$ *satisfies WO, C, and SCTI, if and only if it has a prior-anticipatory representation.*

In order to characterize posterior-anticipatory representations, we consider additional Independence-type conditions.

The first condition is logically independent from both CTI and SCTI. Axiom I_Λ imposes Independence over the set of lotteries that fully resolve in the first stage, Λ. (Since the set of lotteries which resolve fully in the first stage is isomorphic to the set of one-stage lotteries, Independence has a natural interpretation on this sub-domain.)

Independence over Early Resolving Lotteries (I_Λ): *The relation $\succsim_\Lambda$ satisfies Independence.*

The second implies SCTI (and so CTI). It is full two-stage independence; i.e. independence holds when mixing any two compound lotteries in the first stage.

Two-Stage Independence (TI): $P \succsim P'$ *if and only if* $\alpha P + (1-\alpha)Q \succsim \alpha P' + (1-\alpha)Q$.

TI implies that prior beliefs do not matter when considering preferences over information structures. Our nex tresult shows that either TI alone, or SCTI in conjunction with I_Λ, is equivalent to a posterior anticipatory representation.

Proposition 3. *The following are equivalent:*

- *The relation $\succsim$ satisfies WO, C, SCTI, and I_Λ*
- *The relation $\succsim$ satisfies WO, C, and TI*
- *The relation $\succsim$ has a posterior-anticipatory representation*

Special Cases:

The functional forms we have derived above are quite general. In many cases, we may want to suppose further restrictions on the set of functionals we consider.

One typical assumption within the literature is that in either stage, the individual receives the same utility (typically normalized to 0) from beliefs that do not change. We describe these preferences as belief stationarity invariant (BSI).

Definition 4. *An ABB representation is belief-stationarity invariant (BSI) if* $\nu_1(p_i, p_i) = \nu_1(q_i, q_i) = \nu_2(\delta_x, \delta_x) = \nu_2(\delta_y, \delta_y) = 0$ *for all* x, y, p_i, q_i.

A second type of assumption is that the utility derived from beliefs does not depend on the timing of when those beliefs occur. We call this belief time invariance (BTI).[5]

Definition 5. *An ABB representation is belief time invariant (BTI) if $\nu_1 = \nu_2$ over their relevant shared domain.*

[5] BTI is strong because it rules out situations where individuals may get a stronger or weaker "kick" from beliefs if they occur sooner (for example, via discounting). We can weaken BTI to allow for such considerations, and say that BPS representation is pseudo-belief time invariant (PBTI) if for some scalar $\kappa > 0$, $\nu_1 = \kappa\nu_2$ over their relevant shared domain. However, in the end of the proof of Proposition 4 we show that PBTI does not restrict preferences alone and, furthermore, that BSI and PBTI in conjunction have no observable implications as well, as long as κ can be chosen arbitrarily (i.e., is not fixed in a given value).

In order to relate BSI and BTI to behavior, we discuss a certain restriction on preferences over compound lotteries. The next axiom is due to Segal (1990).

Time Neutrality (TN): *If $P \in \Gamma$, $Q \in \Lambda$, and $\phi(P) = \phi(Q)$, then $P \sim Q$.*

Time Neutrality supposes that a decision-maker is indifferent between a lottery that resolves early, or a lottery that resolves late, provided that they induce the same probability distribution over final outcomes. This implies that ordering of preferences is the same over fully early resolving lotteries and fully late resolving lotteries (mapping them to their reduces formed equivalents).

Although BSI and BTI do not restrict preferences alone, in conjunction they do.

Proposition 4. *Suppose $\succsim$ has an ABB representation. The following statements are true:*

1. *The relation $\succsim$ has a representation which is belief stationarity invariant.*
2. *The relation $\succsim$ has a representation which is belief time invariant.*
3. *The relation $\succsim$ has a representation which is both belief stationarity invariant and belief time invariant, if and only if it satisfies Time Neutrality.*

Uniqueness:

The fact that we can obtain either a BSI or a BTI representation without loss of generality raises the question to what extent are ABB preferences uniquely identified. The uniqueness property can be broken up into two parts: First, an immediate application of the mixture space theorem implies that if V and V' are both ABB representations of the same preference relation, then they differ by a positive affine transformation.

Proposition 5. *Suppose $\succsim$ has an ABB representation V. The ABB representation V' also represents $\succsim$ if and only if there exist scalars $\alpha > 0$ and β such that $V' = \alpha V + \beta$.*

Second, there are individual terms that can be subtracted from one component and absorbed in another, leaving the numerical value intact. In Appendix 6.2 we show that the uniqueness results of the sub-components u, ν_1, and ν_2 are more subtle because any outcome that generates material utility must also appear in the support of the beliefs entering ν_1 and ν_2. Thus, one should expect that, without any further restrictions, there is some freedom to assign utility that is generated by any x appearing in the support of the lottery to either material utility or belief-based utility. In particular, this suggests that attitudes towards risk cannot be uniquely identified — attitudes towards final outcomes can be adjusted across all three functions that compose the representation.

If, instead, we focus on the standard normalization applied in the literature (i.e. the one imposed by BSI), then u is unique up to affine transformation, while ν_1 and ν_2 are unique up to common scaling. Because any PBS preferences have a BSI representation, this uniqueness result is entirely general.[6]

Proposition 6. *Suppose* $\succsim$ *has an ABB representation* (u, ν_1, ν_2) *that satisfied BSI. The ABB representation* (u', ν_1', ν_2') *also represents* $\succsim$ *and satisfied BSI if and only if there exist scalars* $\alpha > 0$ *and* β_u *such that*

- $u'(x) = \alpha u + \beta_u$
- $\nu_1'(\rho, p) = \alpha \nu_1(\rho, p)$
- $\nu_2'(p, \delta_x) = \alpha \nu_2(p, \delta_x)$

3 ABB and Recursive Preferences

ABB preferences are not the only preferences used to model decisions over compound risk; an alternative specification is of preferences that are recursive. Recursive preferences have also played an extensive role in a variety of models attempting to capture, among other things, choices over compound lotteries and information (see Kreps and Porteus, 1978; Segal, 1990; Grant, Kajii, and Polak, 1998; Dillenberger, 2010; and Sarver, 2016).

Segal (1990) was the first to formally discuss recursive preferences on the domain of compound lotteries. In the definition below, $\mathbb{CE}_W(p)$ denotes the certainty equivalent of $p \in \Delta$ corresponding to the real function W on Δ, that is, $W(p) = W(\delta_{\mathbb{CE}_W(p)})$.[7]

Definition 6. *Suppose preferences over* Δ^2 *can be represented by the functional* V. *We say that preferences have a recursive representation* (V_1, V_2), *where* $V_i : \Delta \to \mathbb{R}$, *if and only if for all* $P = (p_1, P(p_1); \ldots; p_n; P(p_n))$, *we have* $V(P) = V_1(\mathbb{CE}_{V_2}(p_1), P(p_1); \ldots; \mathbb{CE}_{V_2}(p_n); P(p_n))$.

Segal (1990) provided a behavioral equivalent for these functional forms using an axiom he called Compound Independence, which we refer to as Recursivity.

Recursivity (R): *For any* $p, q \in \Delta$, $Q \in \Delta^2$, *and* $\alpha \in [0,1]$, $D_p \succsim D_q$ *if and only if* $\alpha D_p + (1-\alpha)Q \succsim \alpha D_q + (1-\alpha)Q$.

Recursivity, like CTI, applies Independence to a particular subset of compound lotteries. In particular, the original pair of lotteries being compared must be degenerate in the first stage, that is, members of Γ. Like CTI, Independence is thus applied to a particular "slice" of the compound

[6] We only consider transformations generate different actual values for all sub-functions involved in the transformation, and do not consider transformations that add an subtract elements to a specific sub-functoinal that leave its own actual value unchanged. For example, $\sum_x p(x)\gamma_\nu(p, \delta_x) = \sum_x p(x)(\gamma_\nu(p, \delta_x) + \epsilon_p(x))$ whenever $\sum_x p(x)\epsilon_p(x) = 0$.

[7] For the certainty equivalent to be well-defined, we need to impose some order on the set X. It will be the case whenever we take the set of prizes to be an interval $X \subset \mathbb{R}$ and both functions V_i are monotone with respect to first-order stochastic dominance.

lottery space. However, it is an orthogonal slice to that considered by CTI (or SCTI). Segal (1990) shows that the relation $\succsim$ satisfies WO, C, and R, if and only if it admits a recursive representation.

One immediate question is to what extent these two classes of utility, ABB and recursive, are related. Although they apply Independence to different slices of the compound lottery space, it is unclear whether there exist preferences which have both representations. The next result answers this question in the affirmative, and moreover, shows that the intersection is exactly those preferences which admit a posterior-anticipatory representation.

Proposition 7. *The following are equivalent:*

- *The relation $\succsim$ satisfies WO, C, CTI, and R*
- *The relation $\succsim$ has a posterior-anticipatory representation*
- *The relation $\succsim$ has a recursive representation where V_1 is expected utility*
- *The relation $\succsim$ satisfies WO, C, R, and I_Λ*

Figure 1 depicts the relationships discussed in the last two sections.

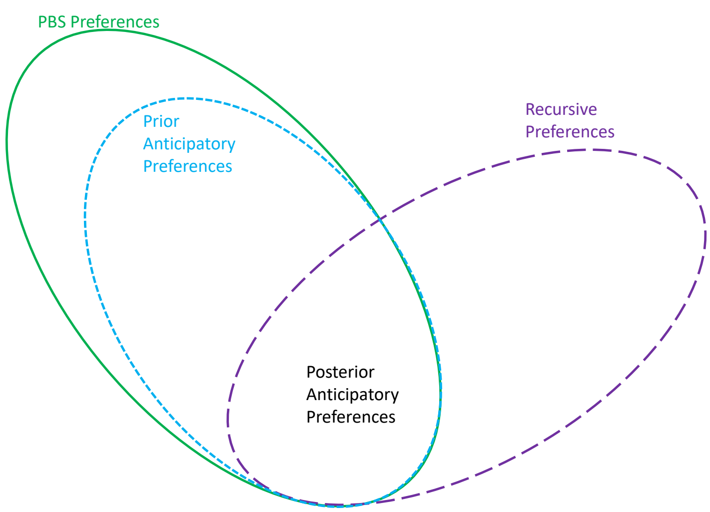

Figure 1: Relationships between Models

In general, ABB models can be directly tested in the most natural domain of information preferences given a fixed prior belief. In contrast, recursive preferences, *as a general class*, have no observable restrictions when the prior belief is fixed; testing the assumption of Recursivity (Axiom R) requires observing preferences as the prior changes. Proposition 7 then implies that this latter property applies as well to the subclass of preferences that have posterior-anticipatory representation (which is another manifestation of Axiom I_Λ). The proposition further implies that any posterior-anticipatory representation captures individuals who have "emotions over emotions". This is due to the fact that recursive models transform any two-stage compound lottery into a simple lottery over the second-stage certainty equivalents. Those certainty equivalents capture all second-period utility from beliefs changing in the second stage (e.g., disappointment/elation). Since first-period preferences take the certainty equivalents as the possible outcomes, those anticipated second period emotions are part and parcel of the "material" payoffs of the first-stage preferences. In all ABB models which do not have posterior-anticipatory representation, future changes in beliefs are independent of the effects of previous changes in beliefs. In other words, first period's beliefs-based utility, as captured by ν_1, is all about changes in material outcomes and does not depend on second period's belief-based utility, ν_2.

In addition to axiom R, Segal (1990) also introduced several other restrictions on preferences over compound lotteries (such as the Reduction of Compound Lotteries axiom and the requirement that Independence holds among both the set of full early resolving lotteries and fully late resolving lotteries). To complete our analysis, in Appendix 6.3 we establish their relationship with CTI, SCTI, and TI, and further interpret these connections via the functional forms.

4 ABB and the Timing of Uncertainty Resolution

Individuals with ABB utility will have intrinsic preferences over information, that is, they may prefer one information structure to another even in the absence of the ability to condition actions on either of them. Many papers looking at specific examples of ABB functional forms derive results regarding preferences over information, while focusing on two concepts: preferences for early versus late resolution of uncertainty and preferences for one-shot versus gradual resolution of uncertainty. In an analogous vein, characterizations of these informational attitudes have been a major focus of the decision-theoretic literature. However, there do not exist equivalent characterizations for ABB preferences. Our results will allow us to compare how different classes of models (recursive and ABB) can accommodate (or not) different non-instrumental attitudes towards information.

Many authors, such as Kreps and Porteus (1978) and Grant, Kajii, and Polak (1998), conjecture that individuals not only prefer uncertainty to be fully resolved earlier (in period 1 rather than in period 2) but also that they always prefer Blackwell-more-informative signals in period 1, that is, earlier resolution of uncertainty. Drawing on Grant, Kajii, and Polak (1998), we can define a preference for early resolution of uncertainty.

Definition 7. *The relation $\succsim$ displays a preference for early resolution of uncertainty (PERU) if for any $Q, P \in \Delta^2$ such that $Q = (q_i, Q(q_i))_{i=1}^n$, $P = ((q_i, Q(q_i))_{i \neq j}; p_1, \beta Q(q_j); p_2, (1-\beta)Q(q_j))$ for some $\beta \in [0,1]$, and $q_j = \beta p_1 + (1-\beta)p_2$, we have $P \succsim Q$.*

Preference for late resolution of uncertainty is analogously defined, by requiring that for the lotteries above $Q \succsim P$.

We now characterize preferences that exhibit preferences for either earlier or later resolved lotteries. Similarly to known results about recursive preferences, attitude towards the resolution of uncertainty is characterized in our model by the curvature of the appropriate components. In the next result we refer to preferences which have a prior-anticipatory representation, but do not have a posterior-anticipatory representation, as having a prior*- anticipatory representation.

Proposition 8. *The following statements are true:*

1. *Suppose $\succsim$ has ABB representation. Then $\succsim$ exhibits a preference for early (resp., late) resolution of uncertainty if and only if $\nu_1(\rho,\cdot)+\sum_x \nu_2(\cdot,x)$ is convex (resp., concave).*

2. *Suppose $\succsim$ has a prior*-anticipatory representation. Then $\succsim$ exhibits a preference for early (resp., late) resolution of uncertainty if and only if $\hat{\nu_2}$ is convex (resp., concave).*

3. *Suppose $\succsim$ has a posterior-anticipatory representation. Then $\succsim$ exhibits a preference for early (resp., late) resolution of uncertainty if and only if $\bar{\nu_1}$ is convex (resp., concave).*

A distinct notion of time preferences is discussed by Dillenberger (2010). He supposes that individuals satisfy Time Neutrality (axiom TN described earlier) and that they prefer either (degenerate) compound lotteries in which all uncertainty is resolved in period 1 or in period 2 (those lotteries in Λ or Γ) to any other compound lotteries which induce the same prior beliefs. He defines this as a preference for one-shot resolution of uncertainty.

Definition 8. *The relation $\succsim$ exhibits a preference for one-shot resolution of uncertainty (PORU) if: (i) $\succsim$ satisfies TN, and (ii) For all $P,Q,R\in\Delta^2$ such that $\phi(P)=\phi(Q)=\phi(R)$, if $P\in\Lambda$ and $Q\in\Gamma$, then $P\sim Q\succsim R$.*

In contrast to Proposition 8, the characterization of PORU for general ABB preferences is not so immediate to interpret. Furthermore, if preferences depend only on the level of beliefs, then they cannot generate strict PORU. In other words, there are no triple as in Definition 8 for which $P\sim Q\succ R$ hold.

Proposition 9. *The following statements are true:*

1. *Suppose $\succsim$ has ABB representation. Then $\succsim$ exhibits a preference for one shot resolution of information if and only if*

$$\sum_x \phi(P)(x)\nu_1(\phi(P),\delta_x) \geq \sum_i P(p_i)\nu_1(\phi(P),p_i)+\sum_{p_i} P(p_i)\sum_x p_i(x)\nu_1(p_i,\delta_x)$$

2. *If $\succsim$ has a prior-anticipatory representation, then it can never exhibit strict PORU.*

For item (1), suppose $\succsim$ has ABB representation. If $\succsim$ exhibit PORU then it must have an ABB representation that satisfies both BSI and BTI, because PORU implies TN. Thus $\nu_1 = \nu_2$. And since the expected utility from material payoffs is the same in all lotteries compared, the result follows.[8] The intuition behind item (2) derives from Corollary 1, where we show that if $\succsim$ has a prior-anticipatory representation, then TN implies that ν_2 is expected utility. Recall that expected utility functionals do not generate any anomalous preferences towards information. The rest of the utility functional depends only on ϕ, the reduced form probability of the lottery. Moreover, since we identify anomalous attitudes towards information by using lotteries which all share the same reduced form probabilities, those other terms always take on the same value, regardless of the information structure. Thus, we can never observe strict PORU. This is in contrast to recursive preferences, where Dillenberger (2010) shows that strict PORU can occur and characterizes it in terms of a single condition on $\succsim_\Gamma = \succsim_\Lambda$.

5 Discussion

We now discuss how our general framework applies to some specific functional forms used in the literature. In order to provide a sense of the breadth of models that our approach encompasses, Table 1 provides a list of papers which use functional forms nested by ABB. Many of these models also allow individuals to take intermediate actions, so their domain and representation may appear different than that presented in this paper.[9]

Some models in the literature fit into the framework of changing beliefs models — they have an ABB representation, but do not have any anticipatory representation. These include models that are explicitly meant to captures utility derived from changing beliefs, such as Kőszegi and Rabin (2009) and Pagel (2015). Ely, Frankel, and Kamenica (2015) also model individuals who care about changes in their beliefs. Their model of surprise has the same structure as our ABB representation, but is formally not within the class of models we study because it is discontinuous. Their model of suspense is similar in spirit, but formally different from ABB models not only due to it's lack of continuity, but also because of a non-linear transformation that is applied to the expected utility from changes in beliefs. In other models, utility may not be derived from changes in beliefs per se. Rather, utility is derived from levels of beliefs, but the function that determines the levels

[8] One example of a function that satisfies PORU is $\nu_1(\rho, p) = -d(\rho, p)$, where d is some standard distance measure on the unit simplex. In this case, the utility loss from beliefs moving equals the total expected distance traveled by beliefs. Indeed, for such function the inequality holds term by term (this is the negative of the triangle inequality) and thus also in expectation. One interpretation for such functional form would be that the agent is simply averse to any changes in beliefs, presumably due to some hidden costs of adjustments (think of an agent who pre-committed to some belief-contingent action, or someone who placed bets on his intermediate beliefs) that are 'large enough', overwhelming any effect of good news.

[9] As in the previous section, we denote models that have prior-anticipatory but not posterior-anticipatory representation as having prior*-anticipatory utility.

Changing Beliefs	Prior*-Anticipatory Utility	Posterior-Anticipatory Utility
Caplin and Eliaz (2003)		Kreps and Porteus (1979)
Mullainathan and Shleifer (2005)		Epstein and Zin (1989)
Kőszegi and Rabin (2009)		Caplin and Leahy (2001)
Kőszegi (2010)		Kőszegi (2003)
Ely, Frankel and Kamenica (2015)		Caplin and Leahy (2004)
Pagel (2016)		Eliaz and Spiegler (2006)
		Eliaz and Schotter (2010)
		Szech and Schweitzer (2016)

Table 1: Some Models Nested by ABB

depends on the prior beliefs. These include the models of Mullainathan and Shleifer (2005), where individuals want to see signals that confirm their priors, as well as Caplin and Eliaz (2003), which takes on the form of a prior-dependent Kreps-Porteus representation. Within our domain, we also capture the model of Kőszegi (2010), who explicitly models expectations (i.e. beliefs) that interact with material payoffs (although his domain also allows for actions and material payoffs in Period 1).

Other models in the literature adhere to the anticipatory-utility framework. In particular, our posterior-anticipatory utility delivers the Caplin and Leahy (2001) representation when applied to our domain (without intermediate actions), and thus all the models that are based on their framework are capture by our model. These include Kőszegi (2003), Caplin and Leahy (2004), Eliaz and Schotter (2010), and Szech and Schweitzer (2016).

We know of no existing models that explicitly capture pure prior-anticipatory motivations, that is, models that admit a prior-anticipatory, but not posterior-anticipatory, representation. As our characterization result shows, this amounts to an implicit assumption in the literature that anticipatory utility regarding the levels of beliefs is always accompanied by assuming the Independence axiom over early resolving lotteries, but not over late resolving lotteries. Descriptively, we believe prior-anticipatory motives are important, as they can accommodate behavior that violates expected utility over early resolving lotteries, in accordance with frequently observed experimental results, such as the Allais paradox.

We conclude by discussing the relationship between our model and several related axiomatizations which are also distinct from the literature on recursive preferences. As previously discussed, our functional form nests that of Caplin and Leahy (2001). Caplin and Leahy provide an axiomatization of their functional form, but take as their domain the set of "psychological lotteries", which include lotteries not just over material outcomes, but also over psychological states (i.e., beliefs). Thus, their domain includes objects (psychological states) which are explicitly not observable, and not directly choosable. Our approach, which confines attention to preferences over compound lotteries, ensures that all our restrictions are stated solely in terms of preferences over observable objects.

Yariv (2002) also provides a characterization of belief-dependent utility, although in a very different domain than ours, and for very different purposes. Her work relies on preferences being expressed over objects where beliefs are independently manipulable across periods. In contrast, our beliefs in period 2 are not independent of those in period 1.

Recently Gul, Natenzon and Pesendorfer (2016) have introduced a model that shares some key features of our model. Although the domain and objects of choice are quite different than ours, there are many key similarities in that both papers consider utility functions where individuals gain utility from beliefs and from material payoffs in a way that is additively separable. However, while we suppose individuals calculate the expectation of a belief-based utility using objective probabilities, Gul, Nautenzon and Pesendorfer (2016) allow for non-additive measures.

6 Appendix

6.1 Proofs

Proof of Lemma 1: By Definition 2, a prior-anticipatory representation is given by $\sum_i P(p_i)$ $\sum_j p_i(x_j)u(x_j) + \nu_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Note that preferences admit such a representation if and only if they can be represented by the functional $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$, for some arbitrary non-expected utility functional $\hat{\nu}_1$. Similarly, by Definition 3, a posterior-anticipatory representation is given by $\sum_i P(p_i) \sum_j p_i(x_j)[u(x_j) + \nu_2(x_j)] + \sum_i P(p_i)\nu_1(p_i)$, which is equivalent to a representation of the form $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. Clearly the second representation is a subset of the first.□

Proof of Proposition 1. We first define a prior-conditional representation as $V_{PC} = \sum_i P(p_i)\nu_{PC}(\phi(P), p_i)$.

Claim 1. *The relation $\succsim$ has a prior-conditional representation if and only if it has an ABB representation.*

Proof of Claim 1. Consider the first two terms in the ABB representation. Observe that the term $\sum_i P(p_i) \sum_j p_i(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i)$. Similarly, any $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i) \sum_j p_i(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i)$.

Consider now the third term in the ABB representation. Note that any $\sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$ can be rewritten as $\sum_i P(p_i)\hat{\nu}_2(p_i)$, since p_i embeds all the $x_j s$ in it's support. Moreover, given any $\sum_i P(p_i)\hat{\nu}_2(p_i)$, we can rewrite it as $\sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$.

Thus preferences have an ABB representation if and only if they can be represented by

$$\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i) + \sum_i P(p_i)\hat{\nu}_2(p_i)$$

Simplifying further, observe that any $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i) + \sum_i P(p_i)\hat{\nu}_2(p_i)$ can be rewritten as $\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i)$; and any $\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i) + \sum_i P(p_i)\hat{\nu}_2(p_i)$. We have just proved that $\succsim$ has a representation of the form V_{ABB} if and only if it has a representation of the form $\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i)$.

We now use our new representation for Claim 2.

Claim 2. *The relation $\succsim$ has a prior-conditional representation if and only if it satisfies WO, C, and CTI.*

Proof of Claim 2. Observe that the prior-conditional representation holds if and only if for any fixed ϕ preferences are expected utility, which is known to be equivalent to the three conditions in the statement of the claim.

This proves the equivalence in the proposition. □

Proof of Proposition 2. First we define a prior-separable representation as $V_{PS} = \nu_{PS1}(\phi(P)) + \sum_i P(p_i)\nu_{PS2}(p_i)$

Claim 3. *The relation $\succsim$ has a prior-separable representation if and only if it has a prior-anticipatory representation.*

Proof of Claim 3. Recall from the proof of Lemma 1 that $\succsim$ has a prior-anticipatory representation if and only if it has a representation $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Note that this is simply the sum of a utility function defined over the reduced lottery and a recursive utility that is expected utility in the first stage.

We now use our new representation for Claim 4.

Claim 4. *The relation $\succsim$ has a prior-separable representation if and only if it satisfies WO, C, and SCTI.*

Proof of Claim 4. It is easy to check that the axioms are necessary for the representation. For sufficiency, observe that SCTI implies CTI, which, in turns, implies that there exists a representation of the form $\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i)$. Moreover, by SCTI, if $\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i) = \sum_i Q(p_i)\widetilde{\nu}(\phi(P), p_i)$, then $\sum_i(\alpha P + (1-\alpha)R)(p_i)\widetilde{\nu}(\phi((\alpha P + (1-\alpha)R)), p_i) = \sum_i(\alpha Q + (1-\alpha)R)(p_i)\widetilde{\nu}(\phi((\alpha P + (1-\alpha)R)), p_i)$ for any R, which is true if and only if $\widetilde{\nu}$ is additively separable in it's first argument:

$\sum_i P(p_i)\widetilde{\nu}(\phi(P), p_i) = \hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. To see this, observe that with n sub-lotteries, the utility function V_{PS} can be thought of as a function of $n+1$ arguments — the n sub-lotteries and the prior beliefs. Since the representation is additively separable,conditional on the prior, preferences must satisfy separability (i.e, preferential independence in Debreu, 1960) across the sub-lotteries (and all subsets of the sub-lotteries). Further observe that SCTI implies that all subsets of the sub-lotteries and the prior also satisfy separability (preferential independence). Thus, by Debreu (1960) (see also Wakker, 1993) the representation must be additively separable in all components.

This proves the equivalence in the proposition. □

Proof of Proposition 3. First we define a prior-separable expected utility representation as $V_{PSEU} = \sum_x \phi(P)(x)\nu_{PSEU1}(x) + \sum_i P(p_i)\nu_{PSEU2}(p_i)$.

Claim 5. *The relation $\succsim$ has a prior-separable expected utility representation if and only if it has a posterior-anticipatory representation.*

Proof of Claim 5. From Lemma 1, $\succsim$ has a posterior-anticipatory representation if and only if it has a representation $\sum_i P(p_i)\sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. This is simply the sum of a an expected utility function defined over the reduced lottery and a recursive utility that is expected utility in the first stage.

We now use the new representation for Claim 6.

Claim 6. *The relation $\succsim$ has a prior-separable expected utility representation if and only if it satisfies WO, C, SCTI, and I_Λ.*

Proof of Claim 6. It is easy to check that any V_{PSEU} representation satisfies SCTI and I_Λ. For the other direction, notice that we have (given SCTI) a representation of the form $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Moreover, I_Λ implies that Independence is satisfied over lotteries in Λ. Observe that within Λ the representation has the form $\hat{\nu}_1(\phi(P)) + \sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$. The second terms is simply an expected utility functional on Λ. Thus, the first term must be expected utility over the reduced form probabilities in order for Independence to be satisfied.

Claim 7. *The relation $\succsim$ has a prior-separable expected utility representation if and only if it satisfies WO, C and TI.*

Observe that by the mixture space theorem, $\succsim$ satisfies WO, C, and TI if and only if it can be represented by the functional $\sum_i P(p_i)\grave{\nu}(p_i)$. Moreover, if preferences can be represented by $\sum_i P(p_i)\grave{\nu}(p_i)$ then clearly they have a prior-separable expected utility representation (where

$\nu_{PSEU1}(x) = 0$). Similarly, any prior anticipatory representation can be written as $\sum_i P(p_i)\grave{\nu}(p_i)$ where $\grave{\nu}(p_i) = \nu_{PSEU2}(p_i) + \sum_x p_i(x)\nu_{PSEU1}(x)$.

This proves the equivalence in the proposition. □

Proof of Proposition 4. We prove each of the statements in order.

- We first show that if $\succsim$ has an ABB representation then it has a BSI representation in a series of two claims.

  Claim 8. *There exists an equivalent representation $(u, \hat{\nu}_1, \hat{\nu}_2)$ which satisfies the condition $\hat{\nu}_1(\rho, \rho) = 0$ for all ρ.*

  Proof of Claim 8. Denote as $N(p)$ the number of elements in the support of p and sum up below only amongst those elements with positive probability. Define: $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) - \nu_1(p, p)$ and $\hat{\nu}_2(p, \delta_x) = \nu_2(p, \delta_x) + \frac{\nu_1(p,p)}{N(p)p(x)}$. By construction, $\hat{\nu}_1(\rho, \rho) = 0$. Moreover, preferences did not change as the new representation gives utility:

$$\sum_x \rho(x)u(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p \sum_x P(p)p(x)\hat{\nu}_2(p, \delta_x)$$

  or

$$\sum_x \rho(x)u(x) + \sum_p P(p)[\nu_1(\rho, p) - \nu_1(p, p)] + \sum_p \sum_x P(p)p(x)[\nu_2(p, \delta_x) + \frac{\nu_1(p, p)}{N(p)p(x)}]$$

  or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\nu_1(\rho, p) - \sum_p P(p)\nu_1(p, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x) + \sum_p P(p)\nu_1(p, p) \sum_x \frac{1}{N(p)}$$

  or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\nu_1(\rho, p) - \sum_p P(p)\nu_1(p, p) + \sum_p P(p)\nu_1(p, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x)$$

  which is the original utility function.□

  Claim 9. *There exists an equivalent representation $(\widetilde{u}, \hat{\nu}_1, \widetilde{\nu}_2)$, which satisfies the condition $\widetilde{\nu}_2(\delta_x, \delta_x) = 0$ for all $x \in X$.*

Proof of Claim 9. Define $\widetilde{\nu}_2(p,\delta_x) = \hat{\nu}_2(p,\delta_x) - \hat{\nu}_2(\delta_x,\delta_x)$ and $\widetilde{u}(x) = u(x) + \hat{\nu}_2(\delta_x,\delta_x)$. Note that $\widetilde{\nu}_2(\delta_x,\delta_x) = 0$ for all x. Observe that this does not change preferences since utility under this representation is:

$$\sum_x \widetilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho,p) + \sum_p \sum_x P(p)p(x)\widetilde{\nu}_2(p,\delta_x)$$

or

$$\sum_x \rho(x)[u(x) + \hat{\nu}_2(\delta_x,\delta_x)] + \sum_p P(p)\hat{\nu}_1(\rho,p) + \sum_p P(p)p(x)[\hat{\nu}_2(p,\delta_x) - \hat{\nu}_2(\delta_x,\delta_x)]$$

or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\hat{\nu}_1(\rho,p) + \sum_p \sum_x P(p)p(x)\hat{\nu}_2(p,\delta_x) + \sum_x \rho(x)\hat{\nu}_2(\delta_x,\delta_x) - \sum_p \sum_x P(p)p(x)\hat{\nu}_2(\delta_x,\delta_x)$$

which is simply the original utility function.□

Thus, we have a utility representation $(\widetilde{u}, \hat{\nu}_1, \widetilde{\nu}_2)$ which satisfied BSI.

- We next show that $\succsim$ always has a representation which is belief-time invariant. Take the representation $(\widetilde{u}, \hat{\nu}_1, \widetilde{\nu}_2)$ defined in the previous part. Define

$$\widetilde{\nu}'_2(p,\delta_x) = \widetilde{\nu}_2(p,\delta_x) + [\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]$$

and

$$\hat{\nu}'_1(\rho,p) = \hat{\nu}_1(\rho,p) - \sum_x p(x)[\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]$$

Observe $(\widetilde{u}, \hat{\nu}'_1, \widetilde{\nu}'_2)$ represents the same preferences. Utility under the second representation is:

$$\sum_x \widetilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}'_1(\rho,p) + \sum_p \sum_x P(p)p(x)\widetilde{\nu}'_2(p,\delta_x)$$

or

$$\sum_x \widetilde{u}(x)\rho(x) + \sum_p P(p)[\hat{\nu}_1(\rho,p) - \sum_x p(x)[\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]]$$
$$+ \quad \sum_p \sum_x P(p)p(x)[\widetilde{\nu}_2(p,\delta_x) + [\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]]$$

or

$$\sum_x \widetilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho,p) - \sum_p \sum_x P(p)p(x)[\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]$$
$$+ \quad \sum_p \sum_x P(p)p(x)\widetilde{\nu}_2(p,\delta_x) + \sum_p \sum_x P(p)p(x)[\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)]$$

or

$$\sum_x \widetilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho,p) + \sum_p \sum_x P(p)p(x)\widetilde{\nu}_2(p,\delta_x)$$

which are the original preferences.

Moreover, observe that by construction

$$\hat{\nu}_1'(p,\delta_x) = \hat{\nu}_1(p,\delta_x) - [\hat{\nu}_1(\delta_x,\delta_x) - \widetilde{\nu}_2(\delta_x,\delta_x)] = \hat{\nu}_1(p,\delta_x) - [0-0]$$

Also

$$\widetilde{\nu}_2'(p,\delta_x) = \widetilde{\nu}_2(p,\delta_x) + [\hat{\nu}_1(p,\delta_x) - \widetilde{\nu}_2(p,\delta_x)] = \hat{\nu}_1(p,\delta_x)$$

Thus we satisfy BTI. However, we no longer satisfy BSI. This is because

$$\hat{\nu}_1'(\rho,\rho) = \hat{\nu}_1(\rho,\rho) - \sum_x \rho(x)[\hat{\nu}_1(\rho,\delta_x) - \widetilde{\nu}_2(\rho,\delta_x)]$$

no longer necessarily equals 0.

- We now show that $\succsim$ has a representation which is both belief-stationary invariant and belief-time invariant if and only if it satisfies TN.

  For the only if part, observe that for $P \in \Gamma$

$$\begin{aligned} V_{ABB}(P) &= E_{\phi(P)}(u) + \nu_1(\phi(P),\phi(P)) + \sum_j \phi(P)(x_j)\nu_2(\phi(P),\delta_{x_j}) \\ &= E_{\phi(P)}(u) + \sum_j \phi(P)(x_j)\nu_2(\phi(P),\delta_{x_j}) \end{aligned}$$

  where the second equality is by BSI.

For $Q \in \Lambda$ we have

$$\begin{aligned} V_{ABB}(Q) &= E_{\phi(Q)}(u) + \sum_j \phi(Q)(x_j)\nu_1(\phi(Q), \delta_{x_j}) + \sum_j \phi(Q)(x_j)\nu_2(\delta_{x_j}, \delta_{x_j}) \\ &= E_{\phi(Q)}(u) + \sum_j \phi(Q)(x_j)\nu_1(\phi(Q), \delta_{x_j}) \end{aligned}$$

where the second equality is again by BSI.

By BTI, $\sum_j \phi(P)(x_j)\nu_2(\phi(P), \delta_{x_j}) = \sum_j \phi(Q)(x_j)\nu_1(\phi(Q), \delta_{x_j})$. And if $\phi(P) = \phi(Q)$, then indeed $V_{ABB}(P) = V_{ABB}(Q)$, that is, TN is satisfied.

To prove the other direction, we can simply assume preferences satisfy BSI. Observe that time neutrality implies that

$$\sum_x \widetilde{u}(x)\rho(x) + \hat{\nu}_1(\rho, \rho) + \sum_x \rho(x)\widetilde{\nu_2}(\rho, \delta_x) = \sum_x \widetilde{u}(x)\rho(x) + \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x) + \sum_x \rho(x)\widetilde{\nu_2}(\delta_x, \delta_x)$$

or, taking the fact that BSI holds

$$\sum_x \rho(x)\widetilde{\nu_2}(\rho, \delta_x) = \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$$

Observe that $\hat{\nu}_1(\rho, \delta_x)$ only appears as a term as part of the sum $\sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$. Thus, we cannot separately identify the individual parts of $\sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$. Since $\sum_x \rho(x)\widetilde{\nu_2}(\rho, \delta_x) = \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$ we can simply suppose without loss of generality that $\rho(x)\widetilde{\nu_2}(\rho, \delta_x) = \rho(x)\hat{\nu}_1(\rho, \delta_x)$ term by term.

- Lastly, as we mention in Footnote 2, we show that if $\succsim$ has an ABB representation, then it has a representation which is both belief-stationary invariant and pseudo-belief-time invariant.[10] First, normalize the representation using claims 8 and 9 so that it satisfies BSI. We then normalize the representation so that BTI holds as in the second part of the proof of this proposition. As we have mentioned there, $\hat{\nu}'_1(\rho, \rho)$ no longer necessarily equals 0. But, since we started with a BSI representation, we already had that $\hat{\nu}'_1(\delta_x, \delta_x) = \widetilde{\nu}'_2(\delta_x, \delta_x) = 0$ so those values do not change.

  In order to simplify notation, call the functionals after these two steps u, ν_1, and ν_2 respectively. Thus, $\nu_1 = \nu_2$ over their shared domain, and $\nu_2(\delta_x, \delta_x) = 0 = \nu_1(\delta_x, \delta_x)$.

[10]Since $\succsim$ always has a representation which is belief-time invariant. This immediately implies that there is also a PBTI representation, where $\kappa = 1$.

Now we will define a representation that satisfies both BSI and PBTI. We do this in a way that mirrors Claim 8. Denote as $N(p)$ the number of elements with positive probability in p and sum up below only amongst those elements. Define: $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) - \nu_1(p, p)$. Importantly, this redefinition implies $\hat{\nu}_1(\rho, \delta_x) = \nu_1(\rho, \delta_x) - \nu_1(\delta_x, \delta_x) = \nu_1(\rho, \delta_x)$.

We then turn to solve for $\hat{\nu_2}$. Denote $z(p) = \nu_1(p, p)$. For our representation to satisfy PBTI we need that $\hat{\nu_2}(p, \delta_x) = \kappa\hat{\nu}_1(p, \delta_x) = \kappa\nu_1(p, \delta_x) = \kappa\nu_2(p, \delta_x)$ for some κ. If p has $N(p)$ outcomes in its support, then these are $N(p)$ equations and $N(p)+1$ unknowns. We also need it to be the case that $\sum p(x)\hat{\nu_2}(p, \delta_x) = z(p)$. Substituting in we get $\kappa \sum p(x)\nu_2(p, \delta_x) = z(p)$ or $\kappa = \frac{z(p)}{\sum p(x)\nu_2(p,\delta_x)}$. Observe that this uniquely pins down κ and so uniquely pins down $\hat{\nu_2}$ for each p. Thus, PBTI is satisfied. Moreover, observe that by construction $\hat{\nu}_2(\delta_x, \delta_x) = 0$ still and $\hat{\nu}_1(\rho, \rho) = 0$, and so BSI is satisfied as well.

□

Proof of Proposition 5. From Claim 1 we know that we can confine attention to a prior-conditional representation of $\succsim$. Observe that fixing $\phi(P)$, $\sum_i P(p_i)\nu_{PC}(\phi(P), p_i)$ is an expected utility functional, and so possesses the same uniqueness results; i.e. it is unique up to affine transformations of scalars $\alpha_P > 0$ and β_P. But, since $\sum_i P(p_i)\nu_{PC}(\phi(P), p_i) \geq \sum_i Q(q_i)\nu_{PC}(\phi(Q), q_i)$ if and only if $\beta_P + \alpha_P \sum_i P(p_i)\nu_{PC}(\phi(P), p_i) \geq \beta_Q + \alpha_Q \sum_i Q(q_i)\nu_{PC}(\phi(Q), q_i)$, it must be the case that $\beta_P = \beta_Q$ and $\alpha_P = \alpha_Q$. □

Proof of Proposition 6. To see the result for a BSI representation, first take the uniqueness result for general ABB preferences (Proposition 10 in Appendix 6.2). Suppose that (u, ν_1, ν_2) is a BSI representation. We first show that any transformation where $\gamma_1(x) \neq 0$ for some x cannot generate a BSI representation. Suppose that there is some x_i such that $\gamma_1(x_i) \neq 0$. Consider the two-stage lottery $D_{\delta_{x_i}}$. Then $\nu_1'(\delta_{x_i}, \delta_{x_i}) = 0 - \gamma_1(x_i) \neq 0$ so this cannot be a BSI representation. Next we show that any transformation where $\gamma_u(x) \neq 0$ for some x cannot generate a BSI representation. Suppose that there is some x_i such that $\gamma_u(x_i) \neq 0$. Consider the two-stage lottery $D_{\delta_{x_i}}$. Then $\nu_2'(\delta_{x_i}, \delta_{x_i}) = 0 - \gamma_u(x_i) \neq 0$ so this cannot be a BSI representation. For similar resasons $\beta_1 = \beta_2 = 0$. Lastly, we show that any transformation where $\gamma_\nu(p, \delta_x) \neq 0$ for some p and x cannot generate a BSI representation. If there were p_i and x_j such that $\gamma_\nu(p_i, \delta_{x_j}) \neq 0$, then $\nu_1'(p, p) = 0 + p(x)\gamma_\nu(p_i, \delta_{x_j}) \neq 0$, violating a BSI representation. □

Proof of Proposition 7. We first show that $\succsim$ has a posterior-separable expected utility representation (i.e. it satisfies WO, C, SCTI, and I_Λ) if and only if it satisfies WO, C, CTI, and R.

Necessity is immediate. To show sufficiency, note that we have a representation of the form $V_{PC} = \sum_i P(p_i)\nu_{PC}(\phi(P), p_i)$. If recursivity is satisfied, then it must be the case that ν_{PC} is independent of the first argument. Thus we have a representation of the form $\sum_i P(p_i)\hat{\nu}_{PC}(p_i)$, which is equivalent to the following representation: $\sum_i P(p_i)\sum_j p_i(x_j)\hat{u}(x_j)+\sum_i P(p_i)\nu_1(p_i)$.

Recall that $\succsim$ has a posterior-anticipatory representation if and only if it has a representation $\sum_i P(p_i)\hat{\nu}(p_i)$. By Segal (1990), this is a recursive representation where V_1 is expected utility. Segal (1990) also shows that this representation holds if and only if $\succsim$ satisfies WO, C, R and I_Λ. □

Proof of Proposition 8. For item (1), observe that we can ignore the first term of the ABB representation, as it is the same under any two compound lotteries with the same reduced form probabilities. Suppose then that $\nu_1(\rho,\cdot)+\sum_x \nu_2(\cdot,x)$ is convex. Then, by Grant, Kajii and Polak (1998) the individual must exhibit a preference for early resolution of uncertainty. Conversely, if the term above is not convex, then it must be concave in a local neighborhood of some p_i. We can replicate the argument in Grant, Kajii and Polak (1998). Take some compound lottery that delivers as one sub-lottery p_i, and take a linear bifurcation of p_i so that the new sub-lotteries are arbitrarily close to p_i. Then by Grant, Kajii and Polak (1998) the individual must be worse off (since locally the utility function is concave).

For item (2), take any P and Q as specified in the statement of the proposition. Observe that $\phi(P)=\phi(Q)$. Direct calculations then show that $P\succsim Q$ if and only if $\beta\hat{\nu}_2(p_1)+(1-\beta)\hat{\nu}_2(p_2)\geq\hat{\nu}_2(\beta p_1+(1-\beta)p_2)$. And since the triple p_1,p_2, and β were arbitrary, the inequality holds if and only if $\hat{\nu}_2$ is convex. Similarly, the inequality is reversed if and only if $\hat{\nu}_2$ is concave.

For item (3), simply replace in the entire paragraph above $\hat{\nu}_2$ with $\bar{\nu}_1$. □

6.2 Uniqueness of ABB representations

In this section we present more detailed uniqueness results.

Proposition 10. *Suppose $\succsim$ has an ABB representation (u,ν_1,ν_2). The ABB representation (u',ν_1',ν_2') also represents $\succsim$ if and only if there exists scalars $\alpha>0,\beta_u,\beta_1,\beta_2$, and continuous functions $\gamma_u: X\to\mathbb{R}$, $\gamma_1: X\to\mathbb{R}$, and $\gamma_\nu:\Delta\times X\to\mathbb{R}$ such that*

- $u'(x)=\alpha u(x)+\beta_u+\gamma_u(x)+\gamma_1(x)$
- $\nu_1'(\rho,p)=\alpha\nu_1(\rho,p)+\beta_1+\sum_x p(x)\gamma_\nu(p,\delta_x)-\sum_x\rho(x)\gamma_1(x)$
- $\nu_2'(p,\delta_x)=\alpha\nu_2(p,\delta_x)+\beta_2-\gamma_\nu(p,\delta_x)-\gamma_u(x)$

Proof of Proposition 10. We first show that if

- $u'(x)=\alpha u+\beta_u+\gamma_u(x)+\gamma_1(x)$
- $\nu_1'(\rho,p)=\alpha\nu_1(\rho,p)+\beta_1+\sum_x p(x)\gamma_\nu(p,\delta_x)-\sum_x\rho(x)\gamma_1(x)$
- $\nu_2'(p,\delta_x)=\alpha\nu_2(p,\delta_x)+\beta_2-\gamma_\nu(p,\delta_x)-\gamma_u(x)$

then (u', ν_1', ν_2') represents the same preferences as (u, ν_1, ν_2).

Consider the utility function generated by the former representation.

$$\sum_x u'\rho(x) + \sum_p P(p)\nu_1'(\rho, p) + \sum_p \sum_x P(p)p(x)\nu_2'(p, \delta_x)$$

or

$$\begin{aligned} & \sum_x \rho(x)[\alpha u + \beta_u + \gamma_u(x) + \gamma_1(x)] \\ + & \sum_p P(p)[\alpha\nu_1(\rho, p) + \beta_1 + \sum_x p(x)\gamma_\nu(p, \delta_x) - \sum_x \rho(x)\gamma_1(x)] \\ + & \sum_p \sum_x P(p)p(x)[\alpha\nu_2(p, \delta_x) + \beta_2 - \gamma_\nu(p, \delta_x) - \gamma_u(x)] \end{aligned}$$

or

$$\begin{aligned} & \alpha \sum_x \rho(x)u + \beta_u + \sum_x \rho(x)\gamma_u(x) + \sum_x \rho(x)\gamma_1(x) \\ + & \alpha \sum_p P(p)\nu_1(\rho, p) + \beta_1 + \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) - \sum_p P(p) \sum_x \rho(x)\gamma_1(x) \\ + & \alpha \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x) + \beta_2 - \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) - \sum_p \sum_x P(p)p(x)\gamma_u(x) \end{aligned}$$

Denoting $\beta = \beta_u + \beta_1 + \beta_2$ and recalling that $\sum_p \sum_x P(p)p(x) = \sum_x \rho(x)$ we get

$$\begin{aligned} & \alpha[\sum_x \rho(x)u + \sum_p P(p)\nu_1(\rho, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x)] + \beta \\ + & \sum_x \rho(x)\gamma_u(x) + \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) + \sum_x \rho(x)\gamma_1(x) \\ - & \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) - \sum_x \rho(x)\gamma_u(x) - \sum_x \rho(x)\gamma_1(x) \end{aligned}$$

or

$$\alpha[\sum_x \rho(x)u + \sum_p P(p)\nu_1(\rho, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x)] + \beta$$

which clearly are the same preferences as (u, ν_1, ν_2).

To prove the other direction, suppose that (u, ν_1, ν_2) and (u', ν_1', ν_2') represent the same preferences.

Define $\hat{u}(x) = u(x) - u(x)$; $\hat{\nu}_2(p, \delta_x) = \nu_2(p, \delta_x) - \nu_2(p, \delta_x)$; and $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) + \sum_x \rho(x)u(x) + \sum_x p(x)\nu_2(p, \delta_x)$. These represent the same preferences as (u, ν_1, ν_2) but we can write $V(P) = \sum_p P(p)\hat{\nu}_1(\phi(P), p)$.

Now define $\hat{u}'(x) = u'(x) - u'(x)$; $\hat{\nu}_2'(p, \delta_x) = \nu_2'(p, \delta_x) - \nu_2'(p, \delta_x)$; and $\hat{\nu}_1'(\rho, p) = \nu_1'(\rho, p) + \sum_x \rho(x)u'(x) + \sum_x p(x)\nu_2'(p, \delta_x)$. These represent the same preferences as (u', ν_1', ν_2') but we can write $V'(P) = \sum_p P(p)\hat{\nu}_1'(\phi(P), p)$.

Since $V(P) = \sum_p P(p)\hat{\nu}_1(\phi(P), p)$ and $V'(P) = \sum_p P(p)\hat{\nu}_1'(\phi(P), p)$ we know that $\hat{\nu}_1'(\phi(P), p)$ must be an affine transformation of $\hat{\nu}_1(\phi(P), p)$; so that $\hat{\nu}_1'(\phi(P), p) = \alpha\hat{\nu}_1(\phi(P), p) + \beta$. Thus $V'(P) = \sum_p P(p)\alpha\hat{\nu}_1(\phi(P), p) + \beta$. Clearly, $\sum_p \alpha P(p)\hat{\nu}_1(\phi(P), p) + \beta$ has an ABB representation $(\alpha u + \beta_u, \alpha\nu_1 + \beta_1, \alpha\nu_2 + \beta_2)$, where $\beta_u + \beta_1 + \beta_2 = \beta$.

By construction $\alpha\hat{u} = \hat{u}' = 0$ and $\alpha\hat{\nu}_2 = \hat{\nu}_2' = 0$. Thus we can say $u'(x) = u'(x) - \alpha u(x) + \alpha u(x)$; $\nu_2'(p, \delta_x) = \nu_2'(p, \delta_x) - \alpha\nu_2(p, \delta_x) + \alpha\nu_2(p, \delta_x)$; and $\nu_1'(\phi(P), p) = \alpha\nu_1(\phi(P), p) - \sum_x \rho(x)[u'(x) - \alpha u(x)] - \sum_x p(x)[\nu_2'(p, \delta_x) - \alpha\nu_2(p, \delta_x)] + \beta$. Moreover, it is easy to verify that we can arbitrarily divide β among the terms.

Define $\gamma_u(x) = 0$; $\gamma_1(x) = u'(x) - \alpha u(x)$; and $\gamma_\nu(p, \delta_x) = -[\nu_2'(p, \delta_x) - \alpha\nu_2(p, \delta_x)]$. Then $u'(x) = \alpha u(x) + \gamma_u(x) + \gamma_1(x)\beta_u$; $\nu_2'(p, \delta_x) = \alpha\nu_2(p, \delta_x) - \gamma_\nu(p, \delta_x) - \gamma_u(x) + \beta_1$ and $\nu_1'(\phi(P), p) = \alpha\nu_1(\phi(P), p) + \sum_x p(x)\gamma_\nu(p, \delta_x) - \sum_x \rho(x)\gamma_1(x)$. Thus we have constructed the transformation.□

For completeness, we now show that if we suppose utility depends only on the levels of beliefs, stronger uniqueness results also obtain. In this case, both belief-based functionals are unique up to expected utility preferences. Thus, the only part of the utility function not uniquely identified (up to standard transformations) are an individual's expected utility attitudes towards final outcomes.

Proposition 11. *Suppose $\succsim$ has an anticipatory representation (u, ν_1, ν_2). The ABB representation (u', ν_1', ν_2') also represents $\succsim$ if and only if there are scalars $\alpha > 0, \beta_u, \beta_1, \beta_2$ and continuous functions $\gamma_u : X \to \mathbb{R}$ and $\gamma_\nu : X \to \mathbb{R}$ such that*

- $u'(x) = \alpha u + \beta_u + \gamma_u(x) + \gamma_\nu(x)$
- $\nu_1'(\rho) = \alpha\nu_1(\rho) + \beta_1 - \sum_x \rho(x)\gamma_\nu(x)$
- $\nu_2'(p) = \alpha\nu_2(p) + \beta_2 - \sum p(x)\gamma_u(x)$[11]

Proof of Proposition 11. The proof is analogous to the one of Proposition 10. We show necessity for prior anticipatory preferences as an example. Consider the utility function generated by the latter representation.

$$\sum_x u'\rho(x) + \sum_p P(p)\nu_1'(\rho) + \sum_p \sum_x P(p)p(x)\nu_2'(p)$$

[11] Observe that in a posterior-anticipatory representation p is a degenerate lottery.

or

$$\sum_x \rho(x)[\alpha u(x) + \beta_u + \gamma_u(x) + \gamma_\nu(x)] + \sum_p P(p)[\alpha \nu_1(\rho) + \beta_1 - \sum_x \rho(x)\gamma_\nu(\delta_x)]$$
$$+ \quad \sum_p \sum_x P(p)p(x)[\alpha \nu_2(p) + \beta_2 - \gamma_u(x)]$$

or

$$\alpha \sum_x \rho(x)u(x) + \beta_u + \sum_x \rho(x)\gamma_u(x) + \sum_x \rho(x)\gamma_\nu(x) + \alpha\nu_1(\rho) + \beta_1 - \sum_x \rho(x)\gamma_\nu(\delta_x)$$
$$+ \quad \sum_p P(p)\alpha\nu_2(p) + \beta_2 - \sum_p \sum_x P(p)p(x)\gamma_u(x)$$

Denoting $\beta = \beta_u + \beta_1 + \beta_2$ and recalling that $\sum_p \sum_x P(p)p(x) = \sum_x \rho(x)$ we get

$$\alpha[\sum_x \rho(x)u(x) + \nu_1(\rho) + \sum_p P(p)\alpha\nu_2(p)] + \beta$$

which clearly are the same preferences as (u, ν_1, ν_2). $\square$

6.3 ABB and Recursive Preferences: Other Properties

In addition to axiom R, Segal (1990) also introduced several other restrictions on preferences over compound lotteries. We first quickly review them. The strongest assumption is called Reduction of Compound Lotteries (ROCL), which supposes that individuals only care about the reduced form probabilities of any given compound lottery.

Reduction of Compound Lotteries (ROCL): *For all $P, Q \in \Delta^2$, if $\phi(P) = \phi(Q)$ then $P \sim Q$.*

Another assumption is to apply Independence not just to the set of full early resolving lotteries (Axiom I_Λ) but also to the set of fully late resolving lotteries. This restriction, which we denote by I_Γ, says that $\succsim_\Gamma$ satisfies Independence.

Independence (I): *Both I_Λ and I_Γ hold.*

A third assumption is **Time Neutrality (TN)**, discussed in the previous section.

Segal (1990), among other things, relates his proposed axioms to one another. In particular, he shows that if $\succsim$ satisfies WO and C, then (*i*) ROCL implies TN; (*ii*) ROCL and R imply I, and ROCL and I imply R; and (*iii*) R, I, and TN, imply ROCL. We can extend Segal's reasoning to include CTI, SCTI, and TI.

Proposition 12. *Suppose $\succsim$ satisfies WO and C. The following statements are true.*[12]

1. *(i) ROCL implies TI; (ii) TI implies SCTI; (iii) SCTI implies CTI.*

2. *(i) R and CTI jointly imply TI (and so SCTI); (ii) SCTI and TN jointly imply ROCL*

3. *TN, R, and CTI jointly imply I (and so ROCL).*

Proof of Proposition 12. We show each part in turn

1. Observe that ROCL implies that all lotteries with the same reduce form probabilities are indifferent, which immediately implies TI. It's clear that TI implies SCTI which implies CTI.

2. R and CTI have been already shown to imply a posterior-anticipatory representation, which implies TI (and so SCTI). SCTI implies that we have a representation of the form $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Over early resolving lotteries this takes the structure $\hat{\nu}_1(\phi(P)) + \sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$, which is simply a non-expected utility functional over the reduced form probabilities. TN implies this must true true also for any lottery with structure $\hat{\nu}_1(\phi(P)) + \nu_2(\phi(P))$, and so $\nu_2(\phi(P)) = sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$, and so ν_2 satisfies reduction, and so ROCL is satisfied.

3. Last, observe that R and CTI imply that SCTI must be satisfied, and we know that SCTI and TN imply ROCL. □

All relationships in Proposition 12 are interpreted via the lens of restrictions on preferences. In the context of our paper, it is perhaps more instructive to interpret them via the functional forms.

Corollary 1. *Suppose $\succsim$ has a prior-anticipatory representation. Then (i) TN or ROCL implies that ν_2 is an expected utility function; and (ii) R or I_Λ implies that $\succsim$ has a posterior-anticipatory representation.*

If $\succsim$ has a posterior-anticipatory representation and satisfies TN, then it is expected utility.

Proof of Corollary 1: A prior-anticipatory representation implies that SCTI is satisfied. From the previous proof we know that SCTI and TN jointly imply ROCL, and that ROCL alone implies TN. Given the representation $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$, TN implies that $\nu_2(p) = sum_i p(\delta_{x_i})\nu_2(\delta_{x_i})$, and so ν_2 is expected utility. If I_Λ or R is satisfied then a posterior-anticipatory representation is implied.

The representation of posterior-anticipatory preferences has the form $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. Observe that over Λ these preferences have the structure $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(\delta_{x_i})\nu_1(\delta_{x_i})$, which is expected utility. Thus, if TN is satisfied, preferences over Γ must also satisfy Independence. □

[12] Items 1(iii) and 2(i) have already been established earlier; we add them here for completeness.

References

[1] Gregory S Berns, Jonathan Chappelow, Milos Cekic, Caroline F Zink, Giuseppe Pagnoni, and Megan E Martin-Skurski. Neurobiological substrates of dread. *Science*, 312(5774):754–758, 2006.

[2] Andrew Caplin and Kfir Eliaz. Aids policy and psychology: A mechanism-design approach. *RAND Journal of Economics*, pages 631–646, 2003.

[3] Andrew Caplin and John Leahy. Psychological expected utility theory and anticipatory feelings. *The Quarterly Journal of Economics*, 116(1):55–79, 2001.

[4] Andrew Caplin and John Leahy. The supply of information by a concerned expert. *The Economic Journal*, 114(497):487–505, 2004.

[5] Gerard Debreu. *Mathematical methods in the social sciences*, chapter Topological methods in cardinal utility theory. Stanford University Press, Stanford, 1960.

[6] David Dillenberger. Preferences for one-shot resolution of uncertainty and Allais-type behavior. *Econometrica*, 78(6):1973–2004, 2010.

[7] Kfir Eliaz and Andrew Schotter. Paying for confidence: An experimental study of the demand for non-instrumental information. *Games and Economic Behavior*, 70(2):304–324, 2010.

[8] Kfir Eliaz and Ran Spiegler. Can anticipatory feelings explain anomalous choices of information sources? *Games and Economic Behavior*, 56(1):87–104, 2006.

[9] Jeffrey Ely, Alexander Frankel, and Emir Kamenica. Suspense and surprise. *Journal of Political Economy*, 123(1):215–260, 2015.

[10] Larry G Epstein and Stanley E Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, pages 937–969, 1989.

[11] Simon Grant, Atsushi Kajii, and Ben Polak. Intrinsic preference for information. *Journal of Economic Theory*, 83(2):233–259, 1998.

[12] Faruk Gul, Paulo Natenzon, and Wolfgang Pesendorfer. Random evolving lotteries and intrinsic preference for information. Technical report, 2016.

[13] Botond Kőszegi and Matthew Rabin. Reference-dependent consumption plans. *The American Economic Review*, 99(3):909–936, 2009.

[14] Botond Kőszegi. Health anxiety and patient behavior. *Journal of health economics*, 22(6):1073–1084, 2003.

[15] David M Kreps and Evan L Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, pages 185–200, 1978.

[16] Richard S Lazarus. Psychological stress and the coping process. 1966.

[17] George Loewenstein. Anticipation and the valuation of delayed consumption. *The Economic Journal*, 97(387):666–684, 1987.

[18] Sendhil Mullainathan and Andrei Shleifer. The market for news. *American Economic Review*, pages 1031–1053, 2005.

[19] Michaela Pagel. A news-utility theory for inattention and delegation in portfolio choice. *Working Paper*, 2016.

[20] Todd Sarver. Optimal reference points and anticipation. Technical report, Discussion Paper, Center for Mathematical Studies in Economics and Management Science, 2012.

[21] Nikolaus Schweizer and Nora Szech. Optimal revelation of life-changing information. Working paper, 2016.

[22] Uzi Segal. Two-stage lotteries without the reduction axiom. *Econometrica*, pages 349–377, 1990.

[23] Peter Wakker. Additive representations on rank-ordered sets: Ii. the topological approach. *Journal of Mathematical Economics*, 22(1):1–26, 1993.

[24] Leeat Yariv. Believe and let believe:axiomatic foundations for belief dependent utility functionals. Technical report, 2001.