A pathwise variation estimate for the sticky particle system

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Abstract

We study collections of finitely many point masses that move freely in space and stick together when they collide via perfectly inelastic collisions. We establish a uniform bound on the mass average of the total variation of the velocities of particle trajectories. This estimate is then employed to reinterpret weak solutions of the sticky particle system

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho v) &= 0 \\
\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) &= 0
\end{align*}
\]

as probability measures on an appropriate path space. In one spatial dimension, we also show that these equations have solutions \( \rho \) and \( v \) satisfying given initial conditions for which \( t \mapsto v(\cdot, t) \# \rho_t \) is a function with values in the 1–Wasserstein space of quantifiably finite variation.

1 Introduction

The sticky particle system (SPS) is a system of PDE that governs the dynamics of a collection of particles that move freely in \( \mathbb{R}^d \) and interact only via perfectly inelastic collisions. Using \( \rho \) to denote the density of particles and \( v \) as an associated local velocity field, the SPS is comprised of the conservation of mass

\[
\partial_t \rho + \nabla \cdot (\rho v) = 0
\]

together with the conservation of momentum

\[
\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = 0.
\]

Both of these equations hold in \( \mathbb{R}^d \times (0, \infty) \). The SPS was first considered in 1970 by Zel’dovich in a model for the expansion of matter without pressure [13]. While this theory stimulated a lot interest in the astronomy community, there is still much to be understood about solutions of SPS.
One of the most fundamental problems regarding the SPS is to find a solution that satisfies a given set of initial conditions. Experience has shown that it makes sense to study this problem aided with the concept a weak solution. In particular, our examples below show that the density $\rho$ will typically be measure–valued and the local velocity $v$ will be discontinuous. As we expect the total mass to be conserved, it makes sense for us to consider the space $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$. We recall this space has a natural topology: $(\sigma_k)_{k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ converges narrowly to $\sigma \in \mathcal{P}(\mathbb{R}^d)$ if

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} gd\sigma_k = \int_{\mathbb{R}^d} gd\sigma$$

for each bounded and continuous $g : \mathbb{R}^d \to \mathbb{R}$.

**Definition 1.1.** Suppose $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable with

$$\int_{\mathbb{R}^d} |v_0| d\rho_0 < \infty.$$

A narrowly continuous $\rho : [0, \infty) \to \mathcal{P}(\mathbb{R}^d); t \mapsto \rho_t$ and Borel measurable $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ is a weak solution of SPS with initial conditions

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad v|_{t=0} = v_0$$

provided the following hold.

(i) For each $T > 0$,

$$\int_0^T \int_{\mathbb{R}^d} |v|^2 d\rho_t dt < \infty.$$

(ii) For each $\psi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$,

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \psi + \nabla \psi \cdot v) d\rho_t dt + \int_{\mathbb{R}^d} \psi(\cdot, 0) d\rho_0 = 0.$$

(iii) For each $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi \cdot v + \nabla \varphi \cdot v) d\rho_t dt + \int_{\mathbb{R}^d} \varphi(\cdot, 0) \cdot v_0 d\rho_0 = 0.$$

**Remark 1.2.** We say that $\rho$ and $v$ is a weak solution of SPS (without making any reference to initial conditions) if (ii) holds for each $\psi \in C_c^\infty(\mathbb{R}^d \times (0, \infty))$ and (iii) holds for $\varphi \in C_c^\infty(\mathbb{R}^d \times (0, \infty); \mathbb{R}^d)$.
In the seminal works of E, Rykov and Sinai [8] and of Brenier and Grenier [3], it was established that there is a weak solution of SPS that satisfies given initial conditions in one spatial dimension ($d = 1$). The more recent paper of Natile and Savaré [11] unifies and builds considerably on these works. We also recommend the paper by Brenier and coauthors [2] which extends [11] by examining more general sticky particle interactions.

In higher spatial dimensions ($d > 1$), much less is known about the existence of solutions to SPS. Nevertheless, there have been some notable works. Sever presented an equation for a flow map in Lagrangian coordinates (equation (3.6) of [12]) whose solutions correspond to weak solutions of SPS; see Dermoune for a very closely related approach involving a stochastic differential equation for a flow map [5, 6]. In [12], Sever also gave a nonconstructive existence proof of his flow map equation for a fairly wide class of initial conditions. However, Bressan and Nguyen have recently shown that Sever’s equation does not have physically reasonable solutions for certain initial conditions in two spatial dimensions [4].

In this paper, we reinterpret weak solutions as Borel probability measures on an appropriate path space. This approach was inspired by the probabilistic interpretation of solutions of the continuity equation described in Chapter 8 of the monograph by Ambrosio, Gigli and Savaré [1]. The main insight we have for these solutions is that the average variation of trajectories is uniformly bounded. This bound implies that weak solutions are compact in a certain sense. We then employ this compactness and an extra entropy estimate available in one spatial dimension to verify the existence of a weak solution that satisfies a particular total variation estimate.

For the remainder of this introduction, we review the concept of a perfectly inelastic collision indicating why it plays a central role in all that follows. We will then state our main results.

1.1 Finite particle systems

Consider $N$ point masses $m_1, \ldots, m_N$ in $\mathbb{R}^d$ with total mass $m_1 + \cdots + m_N = 1$ that move freely unless they collide. When any sub-collection of these particles collide, they stick together to form a particle of larger mass and undergo a perfectly inelastic collision. For example, if masses $m_1, \ldots, m_k$ move with respective velocities $v_1, \ldots, v_k$ before a collision, the new particle that is formed after the collision has mass $m_1 + \cdots + m_k$ and velocity $v$ chosen to satisfy

$$m_1v_1 + \cdots + m_kv_k = (m_1 + \cdots + m_k)v.$$

See Figure 1. In particular, we observe that $v$ is the mass average of the individual velocities $v_1, \ldots, v_k$.

It will be convenient for us to define the *sticky particle trajectories*

$$\gamma_1, \ldots, \gamma_N : [0, \infty) \to \mathbb{R}^d$$

as the piecewise linear trajectories that tracks the location of the respective point masses $m_1, \ldots, m_N$. That is $\gamma_i(t)$ is the location of point mass $m_i$ at time $t \geq 0$; this mass could be
by itself or a part of a larger mass if it has collided with other particles prior to time \( t \). See Figure 2 below. We will use the dot notation “\( \cdot \)” to denote the right derivative

\[
\dot{\zeta}(t) := \lim_{\tau \to 0^+} \frac{\zeta(t + \tau) - \zeta(t)}{\tau}
\]

throughout this paper; we shall see that the right derivatives of \( \gamma_1, \ldots, \gamma_N \) will be prescribed in accordance with the rule of inelastic collisions as discussed above. Moreover, these paths satisfy the sticky particle property: for all \( i, j = 1, \ldots, N \)

\[
\gamma_i(s) = \gamma_j(s) \implies \gamma_i(t) = \gamma_j(t) \quad \text{for all} \quad t \geq s.
\]

(1.1)

For given initial positions and velocities, the corresponding sticky particle trajectories \( \gamma_1, \ldots, \gamma_N \) with prescribed positions and velocities at time 0 can be shown to exist by induction. This and other basic facts are summarized in Proposition 2.1 below.

The local mass density at time \( t \geq 0 \) is given by the following Borel probability measure on \( \mathbb{R}^d \)

\[
\rho_t := \sum_{i=1}^{N} m_i \delta_{\gamma_i(t)} \in \mathcal{P}(\mathbb{R}^d).
\]

A corresponding local velocity field of the system is a Borel mapping \( v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d \) for which

\[
v(x, t) = \dot{\gamma}_i(t) \quad \text{when} \quad x = \gamma_i(t).
\]

We will establish an “averaging property” of \( \rho \) and \( v \) that implies that they are indeed a weak solution of SPS; see subsection 2.3 below.
1.2 Variation estimate

One of the insights we present in this paper is the following inequality. It asserts that the average total variation of the piecewise constant paths $\dot{\gamma}_i$ is uniformly controlled by the maximum distance between $\dot{\gamma}_i(0)$ and $\dot{\gamma}_j(0)$ ($i,j = 1, \ldots, N$). In the following statement and below, we will use the notion of the variation of a mapping $\xi : (0, \infty) \to \mathbb{R}^d$

$$V_0^\infty(\xi) := \sup \left\{ \sum_{i=1}^{n} |\xi(t_i) - \xi(t_{i-1})| : 0 < t_0 < \cdots < t_n < \infty \right\}. \quad (1.2)$$

Lemma 1.3. Assume $\gamma_1, \ldots, \gamma_N$ is a collection of sticky particle trajectories associated with the respective masses $m_1, \ldots, m_N$ (as specified in Definition 2.3). Then

$$\sum_{i=1}^{N} m_i V_0^\infty(\dot{\gamma}_i) \leq 2 \max_{1 \leq i,j \leq N} |\dot{\gamma}_i(0) - \dot{\gamma}_j(0)|. \quad (1.3)$$

When $d = 1$, we have the additional estimate: for $0 < s \leq t < \infty$ and $i,j = 1, \ldots, N$

$$\frac{1}{t} |\gamma_i(t) - \gamma_j(t)| \leq \frac{1}{s} |\gamma_i(s) - \gamma_j(s)|. \quad (1.4)$$

We call this the quantitative sticky particle property as it quantifies (1.1). The estimates (1.3) and (1.4) together imply a certain compactness property of weak solutions arising from finite particle systems. We will use this compactness property to prove the following theorem. We acknowledge that parts (i) and (ii) of this result were previously obtained in [3, 8, 11]. The novelties we offer are in giving a different proof of (i) and (ii) and in the new statement (iii).
Theorem 1.4. Suppose $d = 1$, $\rho_0 \in \mathcal{P}(\mathbb{R})$ and $v_0 : \mathbb{R} \to \mathbb{R}$ is Borel measurable with $v_0|_{\text{supp}(\rho_0)}$ continuous and bounded.

There is a weak solution $\rho$ and $v$ of SPS with initial conditions $\rho|_{t=0} = \rho_0$ and $v|_{t=0} = v_0$. Moreover, this solution has the following properties.

(i) For Lebesgue almost every $t > 0$ and each $x, y \in \text{supp}(\rho_t)$,
\[
(v(x, t) - v(y, t))(x - y) \leq \frac{1}{t}(x - y)^2.
\]

(ii) For any convex $F : \mathbb{R} \to \mathbb{R}$ and Lebesgue almost every $0 \leq s \leq t < \infty$,
\[
\int_{\mathbb{R}} F(v(x, t))d\rho_t(x) \leq \int_{\mathbb{R}} F(v(x, s))d\rho_s(x).
\]

(iii) There is a right continuous $v_#\rho : [0, \infty) \to \mathcal{P}_1(\mathbb{R}); t \mapsto (v_#\rho)_t$ such that
\[
(v_#\rho)_t = v(\cdot, t)_#\rho_t
\]
for Lebesgue almost every $t > 0$ and
\[
V_0^\infty(v_#\rho) \leq 2 \sup_{x, y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)|.
\]

Property (iii) above is an interpretation of (1.3) in the Eulerian variables $\rho$ and $v$. In this statement, we use the notation $f_#\mu \in \mathcal{P}(\mathbb{R})$
\[
(f_#\mu)(A) := \mu(f^{-1}(A))
\]
to denote the push forward of $\mu \in \mathcal{P}(\mathbb{R})$ by a Borel $f : \mathbb{R} \to \mathbb{R}$ and we define
\[
V_0^\infty(v_#\rho) := \sup \left\{ \sum_{i=1}^{n} W_1((v_#\rho)_{t_i}, (v_#\rho)_{t_{i-1}}) : 0 < t_0 < \cdots < t_n < \infty \right\}.
\]

Here we also consider
\[
\mathcal{P}_1(\mathbb{R}) := \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \int_{\mathbb{R}} |x|d\mu(x) < \infty \right\}
\]
edowed with the $1$-Wasserstein metric
\[
W_1(\mu, \nu) := \inf_\pi \int_{\mathbb{R} \times \mathbb{R}} |x - y|d\pi(x, y), \quad \mu, \nu \in \mathcal{P}_1(\mathbb{R}).
\]
This infimum is taken over Borel probability measures $\pi$ on $\mathbb{R} \times \mathbb{R}$ which have first marginal $\mu$ and second marginal $\nu$. The metric space $(\mathcal{P}_1(\mathbb{R}), W_1)$ is typically referred to as the $1$-Wasserstein space.

This paper is organized as follows. In section 2 we study properties of weak solutions when $\rho_0$ is a convex combination of Dirac masses. Then in section 3 we study the total variation of sticky particle trajectories and prove Lemma 1.3. Next, we consider probability measures on a path space that correspond to weak solutions in section 4. Finally, we prove Theorem 1.4 in section 5. We thank Jin Feng, Wilfrid Gangbo, Emanuel Indrei and Zhenfu Wang for engaging in insightful discussions related to this work.
2 Sticky particle trajectories

In this section, we will study the paths \( \gamma_1, \ldots, \gamma_N \) mentioned in the introduction. First we will show such paths exist and that they have the sticky particle property. Then we exhibit weak solutions associated with these paths. We will conclude this section by establishing inequality (1.4), the quantitative version of the sticky particle property when \( d = 1 \).

Proposition 2.1. Suppose \( m_1, \ldots, m_N > 0 \) with \( \sum_{i=1}^{N} m_i = 1 \), \( x_1, \ldots, x_N \in \mathbb{R}^d \) and \( v_1, \ldots, v_N \in \mathbb{R}^d \) are given. There exist piecewise linear paths

\[
\gamma_i : [0, \infty) \to \mathbb{R}^d \quad (i = 1, \ldots, N)
\]

with

\[
\gamma_i(0) = x_i \quad \text{and} \quad \dot{\gamma}_i(0) = v_i,
\]

such that whenever

\[
\gamma_i(t) = \cdots = \gamma_k(t) \neq \gamma_i(t) \quad \text{for } i \notin \{i_1, \ldots, i_k\}
\]

then

\[
\dot{\gamma}_i(t) = \frac{m_i \dot{\gamma}_i(t-)}{m_{i_1} + \cdots + m_{i_k}} \quad (2.1)
\]

for \( j = 1, \ldots, k \).

Proof. Without loss of generality we may consider \( x_1, \ldots, x_N \) distinct and we argue by induction on \( N \). For \( N = 2 \), there are two cases. The first is when \( t \mapsto x_1 + tv_1 \) and \( t \mapsto x_2 + tv_2 \) never intersect. In this scenario, we set

\[
\gamma_i(t) = x_i + tv_i, \quad t \geq 0 \quad (2.2)
\]

for \( i = 1, 2 \). Otherwise, there is a first time \( s > 0 \) where the paths \( t \mapsto x_i + tv_i \) intersect. In this case, we set

\[
\gamma_i(t) := \begin{cases} 
  x_i + tv_i, & t \in [0, s] \\
  z + (t-s)(m_1v_1 + m_2v_2), & t \in [s, \infty)
\end{cases}
\]

where \( z := x_1 + sv_1 = x_2 + sv_2 \).

Now suppose that claim holds for some \( N \geq 2 \) and consider a system with \( N + 1 \) trajectories. If there are no collisions, then we define \( \gamma_i \) by (2.2) for \( i = 1, \ldots, N + 1 \). Suppose otherwise that there is at least one collision between the trajectories. Let \( s \) denote the first time that the trajectories (2.2) intersect, and let us first assume that a single subcollection of trajectories intersect at time \( s \)

\[
z := x_{i_1} + sv_{i_1} = \cdots = x_{i_k} + sv_{i_k} \neq x_i + sv_i \quad \text{for } i \notin \{i_1, \ldots, i_k\}
\]

\((k \geq 2)\). We also set

\[
v = \frac{m_1v_{i_1} + \cdots + m_kv_{i_k}}{m_{i_1} + \cdots + m_{i_k}}.
\]

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Now consider the system of $N + 1 - (k - 1)$ masses $\{m_i\}_{i \neq i_j}$ and $m_i + \cdots + m_k$ with initial positions $\{x_i + sv_i\}_{i \neq i_j}$ and $z$ and initial velocities $\{v_i\}_{i \neq i_j}$ and $v$. By induction, this data gives rise to $N + 1 - (k - 1)$ trajectories $\{\tilde{\gamma}_i\}_{i \neq i_j}$ and $\tilde{\gamma}$ from $[0, \infty) \to \mathbb{R}^d$, respectively, that satisfy the conclusion of this proposition. We then set

$$\gamma_i(t) = \begin{cases} x_i + tv_i, & t \in [0, s] \\ \tilde{\gamma}_i(t - s), & t \in [s, \infty) \end{cases}$$

for $i \neq i_j$ and

$$\gamma_{i_j}(t) = \begin{cases} x_{i_j} + tv_{i_j}, & t \in [0, s] \\ \tilde{\gamma}(t - s), & t \in [s, \infty) \end{cases}$$

for $j = 1, \ldots, k$. It is immediate from construction that this collection of $N + 1$ paths satisfies the desired properties. Finally, we note that a very similar argument can be made in the case that more than one subcollection of $\gamma_1, \ldots, \gamma_{N+1}$ intersect for the first time at $s$. We leave the details to the reader. \hfill $\blacksquare$

**Remark 2.2.** We do not make use of the hypothesis that $\sum_{i=1}^N m_i = 1$. However, we have included this to stay consistent with then standing assumption that the total mass of all the physical systems we consider is equal to 1.

**Definition 2.3.** Any collection of trajectories $\gamma_1, \ldots, \gamma_N : [0, \infty) \to \mathbb{R}^d$ as specified in the conclusion Proposition 2.1 are **sticky particle trajectories** associated with the respective masses $m_1, \ldots, m_N$, initial positions $x_1, \ldots, x_N \in \mathbb{R}^d$ and initial velocities $v_1, \ldots, v_N$.

For the remainder of this section, we suppose that the masses $m_1, \ldots, m_N > 0$ with $\sum_{i=1}^N m_i = 1$, initial positions $x_1, \ldots, x_N \in \mathbb{R}^d$ and initial velocities $v_1, \ldots, v_N \in \mathbb{R}^d$ are given and fixed. We will denote $\gamma_1, \ldots, \gamma_N$ as a corresponding collection of sticky particle trajectories shown to exist above. Let us now verify the sticky particle property, which asserts that once trajectories intersect they coincide thereafter. The physical interpretation of course is that once particles collide, they remain stuck together.

**Corollary 2.4.** Suppose $\gamma_i(s) = \gamma_j(s)$ for some $s \geq 0$ and $i, j \in \{1, \ldots, N\}$. Then

$$\gamma_i(t) = \gamma_j(t)$$

for all $t > s$.

**Proof.** Suppose the assertion is false. Then there are $s \geq 0$ and $i, j \in \{1, \ldots, N\}$ such that $\gamma_i(s) = \gamma_j(s)$, while

$$\gamma_i(t) \neq \gamma_j(t)$$

for some $t > s$. Set

$$\tau := \sup \{ r \in [s, t] : \gamma_i(r) = \gamma_j(r) \}.$$

Clearly $z := \gamma_i(\tau) = \gamma_j(\tau)$, $\tau < t$, and

$$\gamma_i(\tau + \epsilon) \neq \gamma_j(\tau + \epsilon) \quad (2.3)$$
for $\epsilon \in (0, t - \tau)$.

Since $\gamma_i$ and $\gamma_j$ are piecewise linear, there are $w_i, w_j \in \mathbb{R}^d$ such that

$$\gamma_i(t) = z + (t - \tau)w_i \quad \text{and} \quad \gamma_j(t) = z + (t - \tau)w_j$$

for all $t > \tau$ that is sufficiently close to $\tau$. By (2.3), $w_i \neq w_j$ and so

$$\dot{\gamma}_i(t) = w_i \neq w_j = \dot{\gamma}_j(t).$$

However, this is a contradiction to (2.1). □

### 2.1 Eulerian variables

As in the introduction, we set

$$\rho_t := \sum_{i=1}^{N} m_i \delta_{\gamma_i(t)} \in \mathcal{P}(\mathbb{R}^d) \quad (2.4)$$

for each $t \geq 0$. As the paths $\gamma_1, \ldots, \gamma_N$ are continuous, $\rho : t \mapsto \rho_t$ is narrowly continuous. An associated Borel measure $\mu$ on $\mathbb{R}^d \times [0, \infty)$ is given by

$$\int \int_{\mathbb{R}^d \times [0, \infty)} f(x, t) d\mu(x, t) := \sum_{i=1}^{N} m_i \int_{0}^{\infty} f(\gamma_i(t), t) dt$$

for any Borel $f : \mathbb{R}^d \times [0, \infty) \to [0, \infty)$. In particular, observe that the support of $\mu$ is the union of the graphs of $\gamma_1, \ldots, \gamma_N$

$$\text{supp}(\mu) = \bigcup_{i=1}^{N} \{(x, t) \in \mathbb{R}^d \times [0, \infty) : x = \gamma_i(t)\}.$$ 

By construction of the paths $\gamma_1, \ldots, \gamma_N$, the mapping

$$v : \text{supp}(\mu) \to \mathbb{R}^d; \ (\gamma_i(t), t) \mapsto \dot{\gamma}_i(t) \quad (2.5)$$

is well defined. Indeed if $\gamma_i(t) = \gamma_j(t)$, then $\dot{\gamma}_i(t) = \dot{\gamma}_j(t)$. Moreover, for a given Borel $A \subset \mathbb{R}^d$,

$$v^{-1}(A) = \bigcup_{i=1}^{N} \{(x, t) \in \mathbb{R}^d \times [0, \infty) : x = \gamma_i(t) \text{ and } \dot{\gamma}_i(t) \in A\}.$$ 

Since $\dot{\gamma}_i(t) := \lim_{n \to \infty} n(\gamma_i(t + 1/n) - \gamma_i(t))$ for every $t \geq 0$ and $i = 1, \ldots, N$, $v^{-1}(A)$ is a finite union of measurable sets. As a result, $v$ is Borel measurable. Extending this $v$ to be $0 \in \mathbb{R}^d$ on the complement of $\text{supp}(\mu)$, we obtain an extension from $\mathbb{R}^d \times [0, \infty)$ into $\mathbb{R}^d$ that is Borel measurable. So without any loss of generality, we will consider $v$ defined in (2.5) as a Borel mapping from $\mathbb{R}^d \times [0, \infty)$ into $\mathbb{R}^d$. 9
2.2 Averaging property

We now will state and prove an important averaging property of the sticky particle system. This will be used to show that \( \rho \) and \( v \) defined in (2.4) and (2.5), respectively, is a weak solution of SPS.

**Proposition 2.5.** Assume \( g : \mathbb{R}^d \to \mathbb{R}^d \). Then

\[
\sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(s) \tag{2.6}
\]

for \( 0 \leq s \leq t \).

**Proof.** Let \( t_0 = 0 \) and \( 0 < t_1 < \cdots < t_\ell < \infty \) be the collection of first intersection times of the paths \( \gamma_1, \ldots, \gamma_N \). Recall that all \( t \mapsto \dot{\gamma}_i(t) \) are all constant on the intervals \([t_0, t_1), [t_1, t_2), \ldots, [t_{\ell-1}, t_\ell), [t_\ell, \infty)\). Therefore, it suffices to show

\[
\sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_r) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_k) \tag{2.7}
\]

where \( t_r \) is the largest of \( t_0, \ldots, t_\ell \) that is less than \( t \) and \( k = 0, \ldots, r \).

We will prove (2.7) by induction. For \( k = r \), (2.7) is obvious. So we will assume that it holds for some \( k \in \{1, \ldots, r\} \) and then show how this assumption implies the assertion holds for \( k-1 \). At time \( t_k \), let us suppose that one sub-collection \( \{\gamma_{i_j}\}_{j=1}^{n} \subset \{\gamma_i\}_{i=1}^{N} \) of paths intersect for the first time. Observe that these trajectories also coincide at time \( t \geq t_k \); we will call this common path \( \overline{\gamma} : [t_k, \infty) \to \mathbb{R}^d \). At time \( t_k \), we also note that

\[
\dot{\gamma}_{i_j}(t_k) = v_k := \frac{m_{i_1} \dot{\gamma}_{i_1}(t_k^-) + \cdots + m_{i_n} \dot{\gamma}_{i_n}(t_k^-)}{m_{i_1} + \cdots + m_{i_n}}
\]

for \( j = 1, \ldots, n \). Finally, we recall that for \( i \neq i_j \), \( \dot{\gamma}_i(t_k) \) is equal to \( \dot{\gamma}_i(t_{k-1}) \).

Taking these two observations into account and the induction hypothesis, we find

\[
\sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_r) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_k) = \sum_{i \neq i_j} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_k) + \sum_{j=1}^{n} m_{i_j} g(\gamma_{i_j}(t)) \cdot \dot{\gamma}_{i_j}(t_k) = \sum_{i \neq i_j} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_{k-1}) + \left( \sum_{j=1}^{n} m_{i_j} \right) g(\overline{\gamma}(t)) \cdot v_k = \sum_{i \neq i_j} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t_{k-1}) + g(\overline{\gamma}(t)) \cdot v_k
\]
This argument is readily adapted to the case where more than one sub-collection of \( \gamma_1, \ldots, \gamma_N \) intersect for the first time at \( t_k \). Therefore, we conclude (2.7) and consequently (2.6).

**Corollary 2.6.** Suppose \( F : \mathbb{R}^d \to \mathbb{R} \) is a convex function and \( 0 \leq s \leq t \). Then

\[
\sum_{i=1}^{N} m_i F(\dot{\gamma}_i(t)) \leq \sum_{i=1}^{N} m_i F(\dot{\gamma}_i(s)).
\]

**Proof.** By using a routine smoothing argument, we may assume that \( F \in C^1(\mathbb{R}^d) \). In view of the previous proposition

\[
\sum_{i=1}^{N} m_i F(\dot{\gamma}_i(s)) \geq \sum_{i=1}^{N} m_i [F(\dot{\gamma}_i(t)) + \nabla F(\dot{\gamma}_i(t)) \cdot (\dot{\gamma}_i(s) - \dot{\gamma}_i(t))]
\]

\[
= \sum_{i=1}^{N} m_i F(\dot{\gamma}_i(t)) + \sum_{i=1}^{N} m_i \nabla F(\dot{\gamma}_i(t)) \cdot (\dot{\gamma}_i(s) - \dot{\gamma}_i(t))
\]

\[
= \sum_{i=1}^{N} m_i F(\dot{\gamma}_i(t)) + \sum_{i=1}^{N} m_i \nabla F(v(\gamma_i(t), t)) \cdot (\dot{\gamma}_i(s) - \dot{\gamma}_i(t))
\]

\[
= \sum_{i=1}^{N} m_i F(\dot{\gamma}_i(t)).
\]

**Remark 2.7.** In terms of the Eulerian variables \( \rho \) and \( v \), the previous corollary reads

\[
\int_{\mathbb{R}^d} F(v(x,t))d\rho_t(x) \leq \int_{\mathbb{R}^d} F(v(x,s))d\rho_s(x)
\]

for \( s \leq t \). In particular, choosing \( F(y) = \frac{1}{2}|y|^2 \) implies that total kinetic energy

\[
t \mapsto \int_{\mathbb{R}^d} \frac{1}{2}|v(x,t)|^2 d\rho_t(x)
\]

is nonincreasing.
2.3 Weak solution property

Set

\[ \rho_0 := \sum_{i=1}^{N} m_i \delta_{x_i}, \]

and let \( v_0 : \mathbb{R}^d \to \mathbb{R} \) be any Borel measurable mapping such that

\[ v_0(x_i) = v_i \]

for \( i = 1, \ldots, N \). Let us now argue that \( \rho \) and \( v \) defined in (2.4) and (2.5) is indeed a weak solution of SPS with initial conditions \( \rho|_{t=0} = \rho_0 \) and \( v|_{t=0} = v_0 \). This assertion was likely first verified by [8] when \( d = 1 \). The specific argument we give below is inspired by computations performed in the introduction of [5].

Choose \( \psi \in C^\infty_c(\mathbb{R}^d \times [0, \infty)) \) and note

\[
\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \psi + \nabla \psi \cdot v) d\rho dt = \int_0^\infty \sum_{i=1}^{N} m_i [\partial_t \psi(\gamma_i(t), t) + \nabla \psi(\gamma_i(t), t) \cdot v(\gamma_i(t), t)] dt
\]

\[
= \sum_{i=1}^{N} m_i \int_0^\infty [\partial_t \psi(\gamma_i(t), t) + \nabla \psi(\gamma_i(t), t) \cdot v(\gamma_i(t), t)] dt
\]

\[
= \sum_{i=1}^{N} m_i \int_0^\infty [\partial_t \psi(\gamma_i(t), t) + \nabla \psi(\gamma_i(t), t) \cdot \dot{\gamma}_i(t)] dt
\]

\[
= \sum_{i=1}^{N} m_i \int_0^\infty \frac{d}{dt} \psi(\gamma_i(t), t) dt
\]

\[
= - \sum_{i=1}^{N} m_i \psi(\gamma_i(0), 0)
\]

\[
= - \int_{\mathbb{R}^d} \psi(\cdot, 0) d\rho_0.
\]

Next, we will make use of the averaging property. Let \( \varphi \in C^\infty_c(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d) \), and observe

\[
\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi \cdot v + \nabla \varphi \cdot v) d\rho dt =
\]

\[
= \int_0^\infty \sum_{i=1}^{N} m_i v(\gamma_i(t), t) \cdot (\partial_t \varphi(\gamma_i(t), t) + \nabla \varphi(\gamma_i(t), t) v(\gamma_i(t), t)) dt
\]

\[
= \int_0^\infty \sum_{i=1}^{N} m_i \dot{\gamma}_i(t) \cdot (\partial_t \varphi(\gamma_i(t), t) + \nabla \varphi(\gamma_i(t), t) v(\gamma_i(t), t)) dt
\]

\[
= \int_0^\infty \sum_{i=1}^{N} m_i \dot{\gamma}_i(0) \cdot (\partial_t \varphi(\gamma_i(t), t) + \nabla \varphi(\gamma_i(t), t) v(\gamma_i(t), t)) dt
\]
\[
\begin{aligned}
&= \sum_{i=1}^{N} m_i \dot{\gamma}_i(0) \cdot \int_{0}^{\infty} (\partial_t \varphi(\gamma_i(t), t) + \nabla \varphi(\gamma_i(t), t)v(\gamma_i(t), t))dt \\
&= \sum_{i=1}^{N} m_i \dot{\gamma}_i(0) \cdot \int_{0}^{\infty} (\partial_t \varphi(\gamma_i(t), t) + \nabla \varphi(\gamma_i(t), t)\dot{\gamma}_i(t))dt \\
&= \sum_{i=1}^{N} m_i \dot{\gamma}_i(0) \cdot \int_{0}^{\infty} \frac{d}{dt} \varphi(\gamma_i(t), t)dt \\
&= - \sum_{i=1}^{N} m_i \dot{\gamma}_i(0) \cdot \varphi(\gamma_i(0), 0) \\
&= - \sum_{i=1}^{N} m_i v_0(\gamma_i(0)) \cdot \varphi(\gamma_i(0), 0) \\
&= - \int_{\mathbb{R}^d} \varphi(\cdot, 0) \cdot v_0 d\rho_0.
\end{aligned}
\]

### 2.4 Quantitative sticky particle property \(d = 1\)

As mentioned in the introduction, there is an extra estimate to make use of when \(d = 1\). This estimate implies (1.5), which is an analog of the entropy condition in scalar conservation laws. Inequality (1.5) is well known although few authors seemed to exploit it in their study of the global existence of weak solutions. It will play a central role for us.

**Proposition 2.8.** Assume \(d = 1\). For each \(i, j \in \{1, \ldots, N\}\) and \(t > 0\),

\[
(\dot{\gamma}_i(t) - \dot{\gamma}_j(t))(\gamma_i(t) - \gamma_j(t)) \leq \frac{1}{t}(\gamma_i(t) - \gamma_j(t))^2. \tag{2.8}
\]

**Proof.** Set \(t_0 = 0\), suppose \(t_1 < \cdots < t_\ell < \infty\) are the possible first intersection times of the trajectories \(\gamma_1, \ldots, \gamma_N\) and assume that \(t \in [t_{k-1}, t_k]\). Between these times, all trajectories are linear so

\[
\gamma_i(t) = a_i + tw_i \quad \text{and} \quad \gamma_i(t) = a_j + tw_j
\]

for some \(a_i, a_j \in \mathbb{R}\) and \(w_i := \dot{\gamma}_i(t_{k-1}), w_j := \dot{\gamma}_j(t_{k-1})\).

First suppose

\[
(a_i - a_j)(w_i - w_j) < 0. \tag{2.9}
\]

If in addition

\[
t \leq - \frac{(a_i - a_j)(w_i - w_j)}{|w_i - w_j|^2},
\]

then

\[
(w_i - w_j)(a_i + tw_i - (a_j + tw_j)) = (a_i - a_j)(w_i - w_j) + t(w_i - w_j)^2 \leq 0.
\]

Otherwise,

\[
t > - \frac{(a_i - a_j)(w_i - w_j)}{|w_i - w_j|^2}.
\]
However, this would imply that an intersection of $t \mapsto a_i + tw_i$ and $t \mapsto a_i + tw_i$ occurs at time

$$t_k = -\frac{(a_i - a_j)(w_i - w_j)}{|w_i - w_j|^2}.$$ 

In other words, $t > t_k$ contradicting our assumption. As a result,

$$\begin{align*}
(\dot{\gamma}_i(t) - \dot{\gamma}_j(t))(\gamma_i(t) - \gamma_j(t)) &= (\dot{\gamma}_i(t_k - 1) - \dot{\gamma}_j(t_k - 1))(\gamma_i(t) - \gamma_j(t)) \\
&= (w_i - w_j)(a_i + tw_i - (a_j + tw_j)) \\
&\leq 0.
\end{align*}$$

Alternatively, if (2.9) does not hold, then

$$\begin{align*}
(\dot{\gamma}_i(t) - \dot{\gamma}_j(t))(\gamma_i(t) - \gamma_j(t)) &= (w_i - w_j)((a_i - a_j) + t(w_i - w_j)) \\
&= (w_i - w_j)(a_i - a_j) + t(w_i - w_j)^2 \\
&\leq 2(w_i - w_j)(a_i - a_j) + t(w_i - w_j)^2 \\
&\leq \frac{1}{t}(a_i - a_j)^2 + 2(w_i - w_j)(a_i - a_j) + t(w_i - w_j)^2 \\
&= \frac{1}{t}(a_i - a_j + t(w_i - w_j))^2 \\
&= \frac{1}{t}(a_i + tw_i - (a_j + tw_j))^2 \\
&= \frac{1}{t}(\gamma_i(t) - \gamma_j(t))^2.
\end{align*}$$

Thus (2.8) holds for all $t \in [t_{k-1}, t_k]$. It is also not hard to see the argument above implies that (2.8) holds for $[t_{k}, \infty)$, as well.

Remark 2.9. The above estimate is purely kinematic, in the sense that it does not depend in any way on the masses $m_1, \ldots, m_N$.

The following corollary is immediate once we recall that $\text{supp}(\rho_t) = \{\gamma_1(t), \ldots, \gamma_N(t)\}$ and that

$$\dot{\gamma}_i(t) = v(\gamma_i(t), t)$$

for $t > 0$ and $i = 1, \ldots, N$.

Corollary 2.10. Assume $d = 1$ and that $\rho$ and $v$ are defined in (2.4) and (2.5). Then

$$\begin{align*}
(v(x, t) - v(y, t))(x - y) &\leq \frac{1}{t}(x - y)^2
\end{align*}$$

for $t > 0$ and $x, y \in \text{supp}(\rho_t)$.

We also have the following quantitative sticky particle property mentioned in the introduction.
Corollary 2.11. Assume $d = 1$. For each $i, j \in \{1, \ldots, N\}$ and $0 < s \leq t < \infty$,

$$\frac{1}{t} |\gamma_i(t) - \gamma_j(t)| \leq \frac{1}{s} |\gamma_i(s) - \gamma_j(s)|.$$

Proof. We have by direct computation

$$\frac{d}{dt} \frac{1}{2} |\gamma_i(t) - \gamma_j(t)|^2 = (\gamma_i(t) - \gamma_j(t))(\dot{\gamma}_i(t) - \dot{\gamma}_j(t))$$

$$= (\gamma_i(t) - \gamma_j(t))(v(\gamma_i(t), t) - v(\gamma_j(t), t))$$

$$\leq \frac{1}{t} |\gamma_i(t) - \gamma_j(t)|^2$$

for almost every $t > 0$. As a result,

$$\frac{d}{dt} \frac{1}{2t^2} |\gamma_i(t) - \gamma_j(t)|^2 = \frac{1}{t^2} \frac{d}{dt} |\gamma_i(t) - \gamma_j(t)|^2 - \frac{2}{t^3} |\gamma_i(t) - \gamma_j(t)|^2$$

$$\leq \frac{2}{t^3} |\gamma_i(t) - \gamma_j(t)|^2 - \frac{2}{t^3} |\gamma_i(t) - \gamma_j(t)|^2$$

$$= 0.$$

\[\square\]

3 Variation estimate

In this section, we will verify Lemma 1.3. Recall that this lemma asserts inequality (1.3)

$$\sum_{i=1}^N m_i V_0^\infty (\dot{\gamma}_i) \leq 2 \max_{1 \leq i,j \leq N} |v_i - v_j|$$ 

(3.1)

for any collection of sticky particle trajectories $\gamma_1, \ldots, \gamma_N$ associated with the respective masses $m_1, \ldots, m_N$ (such that $\sum_{i=1}^N m_i = 1$) and initial velocities

$$v_i := \dot{\gamma}_i(0) \quad i = 1, \ldots, N.$$

A basic fact that we will use is: if $\xi$ is right continuous and assumes finitely many values, say

$$\xi(t) = \begin{cases} 
  c_0, & t \in (0, s_1) \\
  c_1, & t \in [s_1, s_2) \\
  \vdots \\
  c_{n-1}, & t \in [s_{n-1}, s_n) \\
  c_n, & t \in [s_n, \infty)
\end{cases}$$

for almost every $t > 0$. As a result,
with \( s_1 < s_2 < \cdots < s_n \), then
\[
V_0^\infty(\xi) = \sum_{i=1}^n |c_i - c_{i-1}|.
\]

This comment is relevant as each \( \dot{\gamma}_i \) is right continuous and assumes finitely many values for each \( i = 1, \ldots, N \).

We will compute a few examples before issuing a general proof of the Lemma 1.3.

**Example 3.1.** Consider the case \( N = 2 \) where \( \gamma_1 \) and \( \gamma_2 \) intersect. See Figure 3 for a diagram. Observe that

\[
V_0^\infty(\dot{\gamma}_1) = |v_1 - (m_1 v_1 + m_2 v_2)| = m_2 |v_1 - v_2|
\]

and
\[
V_0^\infty(\dot{\gamma}_2) = |v_2 - (m_1 v_1 + m_2 v_2)| = m_1 |v_1 - v_2|.
\]

Therefore,
\[
m_1 V_0^\infty(\dot{\gamma}_1) + m_2 V_0^\infty(\dot{\gamma}_2) = 2m_1 m_2 |v_1 - v_2| \leq 2 \left( \frac{m_1 + m_2}{2} \right)^2 |v_1 - v_2| = \frac{1}{2} |v_1 - v_2|.
\]

So (3.1) holds, as desired.
Example 3.2. Suppose $N = 3$, $\gamma_2$ and $\gamma_3$ intersect and then the resulting path intersects with $\gamma_1$. This system is illustrated in Figure 4. We have

$$V_0^\infty (\dot{\gamma}_1) = |v_1 - (m_1 v_1 + m_2 v_2 + m_3 v_3)|$$

$$= |m_2 (v_1 - v_2) + m_3 (v_1 - v_3)|$$

$$\leq m_2 |v_1 - v_2| + m_3 |v_1 - v_3|,$$

$$V_0^\infty (\dot{\gamma}_2) = \left| v_2 - \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} \right| + \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} - (m_1 v_1 + m_2 v_2 + m_3 v_3)$$

$$= \frac{m_3}{m_2 + m_3} |v_2 - v_3| + m_1 \left| v_1 - \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} \right|,$$

and

$$V_0^\infty (\dot{\gamma}_3) = \left| v_3 - \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} \right| + \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} - (m_1 v_1 + m_2 v_2 + m_3 v_3)$$
Consequently,

\[
\sum_{i=1}^{3} m_i V_0^\infty (\dot{\gamma}_i) \leq m_1 m_2 |v_1 - v_2| + m_1 m_3 |v_1 - v_3| + \frac{2m_2 m_3}{m_2 + m_3} |v_2 - v_3|
+ m_1 (m_2 + m_3) \left| v_1 - \frac{m_2 v_2 + m_3 v_3}{m_2 + m_3} \right|
\leq m_1 m_2 |v_1 - v_2| + m_1 m_3 |v_1 - v_3| + \frac{2m_2 m_3}{m_2 + m_3} |v_2 - v_3|
+ m_1 m_2 (v_1 - v_2) + m_3 (v_1 - v_3)|
\leq 2m_1 m_2 |v_1 - v_2| + 2m_1 m_3 |v_1 - v_3| + \frac{2m_2 m_3}{m_2 + m_3} |v_2 - v_3|.
\]

In order to verify (3.1), it suffices to show

\[ m_1 m_2 + m_1 m_3 + \frac{m_2 m_3}{m_2 + m_3} \leq 1. \]

Observe that the elementary estimate we used in the previous example applies in this example

\[ \frac{m_2 m_3}{m_2 + m_3} \leq \frac{1}{m_2 + m_3} \left( \frac{m_2 + m_3}{2} \right)^2 = \frac{m_2 + m_3}{4}. \]

Combining with \( m_1 = 1 - (m_2 + m_3) \) gives

\[ m_1 m_2 + m_1 m_3 + \frac{m_2 m_3}{m_2 + m_3} \leq m_1 (m_2 + m_3) + \frac{m_2 + m_3}{4}
\]
\[ = (1 - (m_2 + m_3))(m_2 + m_3) + \frac{m_2 + m_3}{4}
\]
\[ \leq \max_{0 \leq z \leq 1} \left( (1 - z)z + \frac{1}{4} z \right)
\]
\[ = \frac{25}{64}
\]
\[ < 1. \]

**Proof of Lemma 1.3.** We will first establish this assertion in the case that the final first intersection occurs between two trajectories where one these two has not had any prior intersections. We will argue by induction on \( N \) to establish this special case and then show how it implies the general case.

1. When \( N = 2 \), there are two possibilities. If the trajectories do not collide,

\[ \sum_{i=1}^{2} m_i V_0^\infty (\dot{\gamma}_i) = 0. \]
Otherwise they collide and we have by example (3.1) that
\[ \sum_{i=1}^{2} m_i V_0^\infty (\dot{\gamma}_i) \leq \frac{1}{2} |v_1 - v_2|. \]

Therefore, (1.3) holds for \( N = 2 \).

2. We now suppose \( N \geq 2 \) and pursue the special case of the claim for \( N + 1 \). We further suppose that some collisions occur or else (1.3) is trivial. Without any loss of generality, we may assume that the trajectory that doesn’t intersect with any of the other trajectories before the final first intersection time is associated with mass \( m_{N+1} \). We can also assume that the other trajectory involved in the final first intersection has already intersected with trajectories \( \gamma_{i_1}, \ldots, \gamma_{i_k} \) so that its associated mass is \( m_{i_1} + \cdots + m_{i_k} \) and its velocity is
\[ \frac{m_{i_1} v_{i_1} + \cdots + m_{i_k} v_{i_k}}{m_{i_1} + \cdots + m_{i_k}} \]
at the final first collision time.

The average variation of this system is
\[ \sum_{i=1}^{N+1} m_i V_0^\infty (\dot{\gamma}_i) = \sum_{i=1}^{N} m_i V_0^\infty (\dot{\gamma}_i) + m_{N+1} V_0^\infty (\dot{\gamma}_{N+1}) \]
\[ = Q + 2m_{N+1} \left( \sum_{j=1}^{k} m_{i_j} \right) \left| v_{N+1} - \frac{m_{i_1} v_{i_1} + \cdots + m_{i_k} v_{i_k}}{m_{i_1} + \cdots + m_{i_k}} \right|. \]

Here \( Q \) represents the average variation of the trajectories \( \gamma_1, \ldots, \gamma_N \) if the final collision did not occur. Note in particular that we used our assumption about two trajectories colliding at the final first intersection time and employed (3.2) from Example 3.1 in deriving this exact expression.

By the induction hypothesis and an elementary scaling argument,
\[ Q \leq 2 \left( \sum_{i=1}^{N} m_i \right) \max_{1 \leq i, j \leq N} |v_i - v_j|. \]

Also note
\[ \left| v_{N+1} - \frac{m_{i_1} v_{i_1} + \cdots + m_{i_k} v_{i_k}}{m_{i_1} + \cdots + m_{i_k}} \right| = \left| \frac{m_{i_1} (v_{N+1} - v_{i_1}) + \cdots + m_{i_k} (v_{N+1} - v_{i_k})}{m_{i_1} + \cdots + m_{i_k}} \right| \]
\[ \leq \frac{m_{i_1} |v_{N+1} - v_{i_1}| + \cdots + m_{i_k} |v_{N+1} - v_{i_k}|}{m_{i_1} + \cdots + m_{i_k}} \]
\[ \leq \max_{1 \leq j \leq k} |v_{N+1} - v_{i_j}|. \]

Therefore,
\[ \sum_{i=1}^{N+1} m_i V_0^\infty (\dot{\gamma}_i) \leq 2 \left( \sum_{i=1}^{N} m_i \right) \max_{1 \leq i, j \leq N} |v_i - v_j| + 2m_{N+1} \left( \sum_{i=1}^{N} m_i \right) \max_{1 \leq j \leq k} |v_{N+1} - v_{i_j}| \]
\[ \leq 2 \max_{1 \leq i,j \leq N+1} |v_i - v_j| \left\{ \left( \sum_{i=1}^{N} m_i \right) + \left( \sum_{i=1}^{N} m_i \right) \left( 1 - \sum_{i=1}^{N} m_i \right) \right\} \]
\[ \leq 2 \max_{1 \leq i,j \leq N+1} |v_i - v_j| \left( \max_{0 \leq z \leq 1} \{ z + z(1-z) \} \right) \]
\[ = 2 \max_{1 \leq i,j \leq N+1} |v_i - v_j|, \]
as desired.

3. We will now how show the special case can be used to establish the general case. Let \( \gamma_1, \ldots, \gamma_N \) be \( N \) sticky particle trajectories. Choose \( T > 0 \) so large that no collision occurs after time \( t = T \). Also select \( R > 0 \) so large that
\[ |\gamma_i(t)| < R, \quad 0 \leq t \leq T \]
for each \( i = 1, \ldots, N \). We will need the following technical claim.

Claim 3.3. Fix \( \epsilon > 0 \). There is \( \tau > 0 \) and a linear path
\[ \gamma_{N+1} : [0, \tau] \to \mathbb{R}^d \]
such that

(i) \( \gamma_{N+1} \) does not intersect \( \gamma_1, \ldots, \gamma_N \) on \( [0, \tau] \), and

(ii) there is \( j \in \{1, \ldots, N\} \) for which \( \gamma_{N+1}(\tau) = \gamma_j(\tau) \) and
\[ |\dot{\gamma}_{N+1}(0) - \dot{\gamma}_j(\tau)| \leq \epsilon. \]

Let us assume the claim holds and show to verify inequality (3.1); we will establish the claim at the conclusion of this proof. Fix \( \epsilon > 0 \) and choose \( \gamma_{N+1} \) as above. Select \( N + 1 \) sticky particle trajectories
\[ \gamma_1^\epsilon, \ldots, \gamma_{N+1}^\epsilon : [0, \infty) \to \mathbb{R}^d \]
associated with respective masses
\[ m_i^\epsilon := \frac{m_i}{1 + \epsilon} \quad \text{(for } i = 1, \ldots, N \text{)} \quad \text{and} \quad m_{N+1}^\epsilon := \frac{\epsilon}{1 + \epsilon}, \]
initial positions and initial velocities
\[ \gamma_i^\epsilon(0) = \gamma_i(0) \quad \text{and} \quad \dot{\gamma}_i^\epsilon(0) = \dot{\gamma}_i(0) \]
for \( i = 1, \ldots, N + 1 \).

By design, \( \gamma_{N+1}^\epsilon \) does not intersect with \( \gamma_1^\epsilon, \ldots, \gamma_N^\epsilon \) on \( [0, \tau] \) and has its first intersection with another trajectory at time \( \tau \). Let’s assume this other trajectory is \( \gamma_i^\epsilon \). Just prior to
time $\tau$, $\gamma_i^\epsilon$ could have intersected with various other trajectories $\gamma_i^\epsilon, \ldots, \gamma_i^k$. In this case, $\gamma_i^\epsilon$ is associated with the mass $m_{i_1}^\epsilon + \cdots + m_{i_k}^\epsilon$ and with velocity

$$m_{i_1}^\epsilon v_{i_1} + \cdots + m_{i_k}^\epsilon v_{i_k} = m_{i_1} v_{i_1} + \cdots + m_{i_k} v_{i_k}$$

just prior to $\tau$. By part 2 of this proof,

$$\sum_{i=1}^{N+1} m_i V_0^\infty (\dot{\gamma}_i^\epsilon) = \sum_{i=1}^{N} \frac{m_i}{1 + \epsilon} V_0^\infty (\dot{\gamma}_i^\epsilon) + \frac{\epsilon}{1 + \epsilon} V_0^\infty (\dot{\gamma}_{N+1}^\epsilon) \leq 2 \max \left\{ \max_{1 \leq i,j \leq N} |v_i - v_j|, \max_{1 \leq i \leq N} |\dot{\gamma}_{N+1}(0) - v_i| \right\}. \quad (3.3)$$

In view of Claim 3.3,

$$|\dot{\gamma}_{N+1}(0) - \frac{m_i v_{i_1} + \cdots + m_i v_{i_k}}{m_{i_1} + \cdots + m_{i_k}}| \leq \epsilon.$$ 

As a result,

$$\max_{1 \leq i \leq N} |\dot{\gamma}_{N+1}(0) - v_j| \leq \max_{1 \leq i,j \leq N} |v_i - v_j| + \epsilon$$

and combining with (3.3) gives

$$\sum_{i=1}^{N} \frac{m_i}{1 + \epsilon} V_0^\infty (\dot{\gamma}_i^\epsilon) + \frac{\epsilon}{1 + \epsilon} V_0^\infty (\dot{\gamma}_{N+1}^\epsilon) \leq 2 \max_{1 \leq i,j \leq N} |v_i - v_j| + 2\epsilon.$$

Consequently,

$$\sum_{i=1}^{N} m_i V_0^\infty (\dot{\gamma}_i^\epsilon) \leq 2(1 + \epsilon) \max_{1 \leq i,j \leq N} |v_i - v_j| + 2\epsilon(1 + \epsilon).$$

We can now send $\epsilon \to 0^+$ to conclude (3.1). \hfill \Box

Proof of Claim 3.3. First suppose that all of the $\gamma_i$ are constant after time $T$ (as in Figure 3). That is $\gamma_i(t) =: p_i$ for $t \geq T$ for $i = 1, \ldots, N$. Select $j \in \{1, \ldots, N\}$ so that

$$|p_j| = \max_{1 \leq i \leq N} |p_i|.$$ 

Let us assume initially that $|p_j| > 0$.

For $\tau > T$ and $t \in [0, \tau]$, set

$$\gamma_{N+1}(t) := \left( \frac{t - T}{\tau - T} \right) p_j + \left( \frac{t - \tau}{T - \tau} \right) R \frac{p_j}{|p_j|}.$$
Figure 5: A schematic of the first case we consider in the proof of Claim 3.3 ($d = 1$).

Note that for $0 \leq t \leq T$,

$$|\gamma_{N+1}(t)| = \left(\frac{t - T}{\tau - T}\right) |p_j| + \left(\frac{t - \tau}{T - \tau}\right) R \geq \left(\frac{t - T}{\tau - T}\right) R + \left(\frac{t - \tau}{T - \tau}\right) R = R$$

and for $T \leq t < \tau$,

$$|\gamma_{N+1}(t)| = \left(\frac{t - T}{\tau - T}\right) |p_j| + \left(\frac{t - \tau}{T - \tau}\right) R > \left(\frac{t - T}{\tau - T}\right) |p_j| + \left(\frac{t - \tau}{T - \tau}\right) |p_j| = |p_j|$$

since $|p_j| < R$. Thus, $\gamma_{N+1}$ satisfies (i).

Observe $\gamma_{N+1}(\tau) = p_j$. We also have

$$|\dot{\gamma}_{N+1}(0) - \dot{\gamma}_{i}(\tau)| = |\dot{\gamma}_{N+1}(0)| = \left|\left(\frac{1}{\tau - T}\right) p_j + \left(\frac{1}{T - \tau}\right) R \frac{p_j}{|p_j|}\right|$$

$$= \left|\left(\frac{1}{T - \tau}\right) \left(R \frac{p_j}{|p_j|} - p_j\right)\right|$$

$$= \frac{R - |p_j|}{\tau - T}$$

$$\leq \frac{R}{\tau - T}$$

$$< \epsilon,$$
once we select
\[ \tau > T + \frac{R}{\epsilon}. \]
This proves (ii). It is routine to check that if \(|p_j| = 0\), we can repeat the above computations with
\[ \gamma_{N+1}(t) := \frac{t - \tau}{T - \tau} Rz \]
for any \(z \in \mathbb{R}^d\) with \(|z| = 1\). We leave the details to the reader.

Now let us suppose that not all the trajectories are constant once we pass the final first intersection time \(T\) (as in Figure 6). Then at least one of these trajectories leaves \(B_R(0)\) for all large enough times greater than \(T\). So for \(\tau > T\) chosen large enough, we can select \(j\) such that
\[ |\gamma_j(\tau)| = \max_{1 \leq i \leq N} |\gamma_i(\tau)| \geq R. \]
For this \(\tau\) and \(j\) we also set
\[ \nu := \frac{\gamma_j(\tau)}{|\gamma_j(\tau)|} \]
and
\[ \gamma_{N+1}(t) := R\nu + \frac{t}{\tau}(\gamma_j(\tau) - R\nu), \quad 0 \leq t \leq \tau. \]

Note that \(\gamma_{N+1}(\tau) = \gamma_j(\tau)\). We also have that
\[ \gamma_j(t) = p + (t - T)w, \quad t \geq T \]

Figure 6: A schematic of the second case we consider in the proof of Claim 3.3 \((d = 1)\).

for some $|p| \leq R$ and $|w| \leq \max_{1 \leq i \leq N} |v_i|$ since there are no collisions after time $T$. Therefore,

$$|\gamma_{N+1}(0) - \dot{\gamma}_j(\tau)| = \left| \frac{\gamma_j(\tau) - R\nu}{\tau} - \dot{\gamma}_j(\tau) \right|$$

$$= \left| \frac{p + (\tau - T)w - R\nu}{\tau} - w \right|$$

$$= \left| \frac{p - Tw - R\nu}{\tau} \right|$$

$$\leq \frac{2R + T \max_{1 \leq i \leq N} |v_i|}{\tau}$$

$$< \epsilon$$

for $\tau$ chosen sufficiently large. So $\gamma_{N+1}$ satisfies $(ii)$. Observe, in particular that this choice of $\tau$ is independent of $j$.

As for $(i)$, we first note that

$$|\gamma_{N+1}(t)|^2 = R^2 + \frac{t^2}{\tau^2} |\gamma_i(\tau) - R\nu|^2 + 2 \frac{t}{\tau} R\nu \cdot (\gamma_i(\tau) - R\nu).$$

Since $|\gamma_j(\tau)| \geq R$,

$$\nu \cdot (\gamma_i(\tau) - R\nu) = |\gamma_i(\tau)| - R \geq 0.$$ 

Thus, $|\gamma_{N+1}(t)| \geq R$ for all $t \in [0, \tau]$ and so $\gamma_{N+1}$ does not intersect any of the other paths on the interval $[0, T]$. We claim that

$$\gamma_i(t) \cdot \nu < \gamma_{N+1}(t) \cdot \nu \quad t \in (T, \tau) \quad (3.4)$$

for each $i = 1, \ldots, N$, which would complete our proof of $(ii)$.

In order to verify $(3.4)$, we fix $i \in \{1, \ldots, N\}$ and note

$$\gamma_i(T) \cdot \nu \leq |\gamma_i(T)| < R \leq |\gamma_{N+1}(T)| = \gamma_{N+1}(T) \cdot \nu$$

and

$$\gamma_i(\tau) \cdot \nu \leq |\gamma_i(\tau)| \leq |\gamma_j(\tau)| = \gamma_{N+1}(\tau) \cdot \nu.$$ 

Since

$$[T, \tau] \ni t \mapsto \gamma_i(t) \cdot \nu - \gamma_{N+1}(t) \cdot \nu$$

is an affine function that is negative at $T$ and nonpositive $\tau$, it must be negative for each $t \in (T, \tau)$. As result, $(3.4)$ holds and we conclude. \qed

### 4 Probability measures on the path space

Assume $m_1, \ldots, m_N > 0$ with $\sum_{i=1}^N m_i = 1$ and $x_1, \ldots, x_N \in \mathbb{R}^d$. Set

$$\rho_0 := \sum_{i=1}^N m_i \delta_{x_i},$$

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and let \( v_0 : \mathbb{R}^d \to \mathbb{R}^d \) be a Borel measurable function. By Proposition 2.1, there is a collection of sticky particle trajectories \( \gamma_1, \ldots, \gamma_N \) associated with masses \( m_1, \ldots, m_N > 0 \) initial positions \( x_1, \ldots, x_N \in \mathbb{R}^d \) and velocities

\[
\dot{\gamma}_i(0) = v_0(x_i)
\]

for \( i = 1, \ldots, N \).

Observe that each \( \gamma_i \) belongs to the path space

\[
\Gamma := \{ \gamma : [0, \infty) \to \mathbb{R}^d | \gamma \text{ locally absolutely continuous, } \dot{\gamma} \in L^2_{\text{loc}}((0, \infty); \mathbb{R}^d) \}
\]

It is routine to verify that \( \Gamma \) is a complete, separable metric space when equipped with the distance

\[
d(\gamma, \zeta) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \left[ \max_{0 \leq t \leq n} |\gamma(t) - \zeta(t)| + \left( \int_0^n |\dot{\gamma}(t) - \dot{\zeta}(t)|^2 dt \right)^{1/2} \right].
\]

In particular, \( \lim_{k \to \infty} d(\zeta^k, \zeta) = 0 \) if and only if \( \zeta^k \to \zeta \) locally uniformly on \( [0, \infty) \) and \( \dot{\zeta}^k \to \dot{\zeta} \) in \( L^2_{\text{loc}}((0, \infty); \mathbb{R}^d) \). It will be useful for us to employ the evaluation map

\[
e_t : \Gamma \to \mathbb{R}^d; \gamma \mapsto \gamma(t).
\]

Note that \( e_t \) is continuous for every \( t \geq 0 \).

Let us now introduce the following Borel probability measure on \( \Gamma \)

\[
\eta := \sum_{i=1}^N m_i \delta_{\gamma_i} \in \mathcal{P}(\Gamma).
\]

Recall that in subsection 2.1 we showed \( \gamma_1, \ldots, \gamma_N \) gives rise to a weak solution \( \rho \) and \( v \) defined in (2.4) and (2.5), respectively, that satisfy the initial conditions \( \rho|_{t=0} = \rho_0 \) and \( v|_{t=0} = v_0 \). In terms of this \( \rho \), we have

\[
\rho_t = e_t \# \eta
\]

for each \( t \geq 0 \). That is,

\[
\int_{\mathbb{R}^d} f(x) d\rho_t(x) = \int_{\Gamma} f(\gamma(t)) d\eta(\gamma)
\]

for Borel \( f : \mathbb{R}^d \to [0, \infty) \). Moreover, \( \eta \) is concentrated on paths \( \gamma \) which satisfy the ODE

\[
\dot{\gamma}(t) = v(\gamma(t), t) \quad \text{a.e. } t > 0.
\]

It turns out that any \( \eta \in \mathcal{P}(\Gamma) \) which is concentrated on paths that satisfy (4.3) for some Borel \( v \) generates a solution of the continuity equation by using formula (4.2) (Chapter 8 of [1]). The particular \( \eta \) defined in (4.1) has several other interesting properties that we will discuss below.
4.1 Basic properties

Let us first argue that $\eta$ defined in (4.2) inherits a conservation of momentum property as follows.

**Proposition 4.1.** For each $\varphi \in C_0^\infty(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d)$
\[
\int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \cdot \dot{\gamma}(t)) \, d\eta(\gamma) \, dt + \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) \, d\eta(\gamma) = 0.
\]

**Proof.** Since $\rho$ and $v$ are a weak solution, we can use (4.2) and (4.3) to find
\[
0 = \int_0^\infty \int_\mathbb{R}^d (\partial_t \varphi \cdot v + \nabla \varphi \cdot v) \, d\rho \, dt + \int_\mathbb{R}^d \varphi(\cdot, 0) \cdot v_0 \, d\rho_0
\]
\[
= \int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot v(\gamma(t), t) + \nabla \varphi(\gamma(t), t) v(\gamma(t), t) \cdot v(\gamma(t), t)) \, d\eta(\gamma) \, dt + \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) \, d\eta(\gamma)
\]
\[
= \int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \cdot \dot{\gamma}(t)) \, d\eta(\gamma) \, dt + \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) \, d\eta(\gamma).
\]

For each $t \geq 0$, we will also consider the family of subsets of $\Gamma$
\[
\mathcal{E}(t) := \{ \{ \gamma \in \text{supp}(\eta) : \gamma(t) \in A \} \mid A \subset \mathbb{R}^d \text{ Borel} \}.
\]

This collection is easily seen to be a sub-sigma-algebra of the Borel sigma-algebra on $\Gamma$. In fact, it is the sigma-algebra generated by the mapping $e_t : \text{supp}(\eta) \to \mathbb{R}^d$. These families turn out to have a nice monotonicity property.

**Proposition 4.2.** For each $0 \leq s \leq t < \infty$,
\[
\mathcal{E}(t) \subset \mathcal{E}(s). \tag{4.4}
\]

Before proving this assertion, let’s study the most simplest of examples.

**Example 4.3.** Suppose $N = 2$ and $\tau > 0$ is the first time that $\gamma_1$ and $\gamma_2$ intersect. Here
\[
\gamma = m_1 \delta_{\gamma_1} + m_2 \delta_{\gamma_2},
\]
so that $\text{supp}(\eta) = \{ \gamma_1, \gamma_2 \}$. It follows that for each $A \subset \mathbb{R}^d$ and $t \geq 0$,
\[
\{ \gamma \in \text{supp}(\eta) : \gamma(t) \in A \} = \begin{cases} 
\emptyset, & \text{if } \gamma_1(t) \not\in A, \gamma_2(t) \not\in A \\
\{ \gamma_1 \}, & \text{if } \gamma_1(t) \in A, \gamma_2(t) \not\in A \\
\{ \gamma_2 \}, & \text{if } \gamma_1(t) \not\in A, \gamma_2(t) \in A \\
\{ \gamma_1, \gamma_2 \}, & \text{if } \gamma_1(t) \in A, \gamma_2(t) \in A.
\end{cases}
\]

\[
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\]
Since \( \gamma_1(t) \neq \gamma_2(t) \) for \( t \in [0, \tau) \) and \( \gamma_1(t) = \gamma_2(t) \) for \( t \in [\tau, \infty) \),

\[
\mathcal{E}(t) = \begin{cases} 
\{ \emptyset, \{ \gamma_1 \}, \{ \gamma_2 \}, \{ \gamma_1, \gamma_2 \} \}, & t \in [0, \tau) \\
\{ \emptyset, \{ \gamma_1, \gamma_2 \} \}, & t \in [\tau, \infty) 
\end{cases}
\]

It is now plain to see that (4.4) holds in this example.

**Proof of Proposition 4.2.** Since \( \text{supp}(\eta) = \{ \gamma_1, \ldots, \gamma_N \} \), \( \mathcal{E}(t) \) is equal to the collection of subsets

\[\{ \gamma \in \text{supp}(\eta) : \gamma(t) \in \{ \gamma_{i_1}(t), \ldots, \gamma_{i_k}(t) \} \},\]

where \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\} \). As a result, the more distinct elements of \( \gamma_1(t), \ldots, \gamma_N(t) \) there are, the larger \( \mathcal{E}(t) \) is. The inclusion (4.4) follows because the number of distinct elements of \( \gamma_1(t), \ldots, \gamma_N(t) \) is nonincreasing as time increases. \( \square \)

**Corollary 4.4.** For each \( 0 \leq s \leq t < \infty \), there is a Borel mapping \( f_{t,s} : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[ e_t = f_{t,s} \circ e_s \]

on \( \text{supp}(\eta) \).

**Proof.** We have that \( \mathcal{E}(t) \) is the sigma-algebra generated by \( e_t|_{\text{supp}(\eta)} \) and that \( \mathcal{E}(s) \) is the sigma-algebra generated by \( e_s|_{\text{supp}(\eta)} \). Since \( \mathcal{E}(t) \subset \mathcal{E}(s) \), \( e_t|_{\text{supp}(\eta)} \) is necessarily \( \mathcal{E}(s) \) measurable. It then follows that \( e_t|_{\text{supp}(\eta)} \) is equal to the composition of a Borel measurable mapping \( f_{t,s} : \mathbb{R}^d \to \mathbb{R}^d \) and \( e_s|_{\text{supp}(\eta)} \) (see section 1.3 of [7]). \( \square \)

**Corollary 4.5.** For each \( 0 \leq s \leq t < \infty \),

\[ v(e_t, t) = \mathbb{E}_\eta[v(e_s, s)|e_t]. \]

That is, \( (v(e_t, t)|_{\text{supp}(\eta)})_{t \geq 0} \) is a backwards martingale with respect to filtration \( (\mathcal{E}(t))_{t \geq 0} \).

**Proof.** The assertion follows from Proposition 4.2 and the averaging property (2.6): for each bounded Borel \( g : \mathbb{R}^d \to \mathbb{R}^d \).

\[
\int_{\Gamma} g(\gamma(t)) \cdot v(\gamma(t), t) d\eta(\gamma) = \int_{\Gamma} g(\gamma(t)) \cdot v(\gamma(s), s) d\eta(\gamma).
\]

Moreover, for each \( t \geq 0 \), \( v(e_t, t)|_{\text{supp}(\eta)} \) is \( \mathcal{E}(t) \) measurable as this sigma-algebra is generated by \( e_t|_{\text{supp}(\eta)} \). \( \square \)

### 4.2 Average variation bound

We will now employ a more general notion of variation than the one presented in the introduction (1.2). This notion was developed in the monograph on measure theory by Evans.
and Gariepy [9] and naturally applies to locally integrable mappings. We recall that if \( \xi \in L^1_{\text{loc}}((0, \infty); \mathbb{R}^d) \), the subset of \((0, \infty)\)

\[
L_\xi := \left\{ t > 0 : \lim_{\tau \to 0^+} \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} |\xi(s) - \xi(t)| \, ds = 0 \right\}
\]

has full Lebesgue measure in \((0, \infty)\); this is a consequence of the Lebesgue differentiation theorem.

**Definition 4.6.** Let \( \xi \in L^1_{\text{loc}}((0, \infty); \mathbb{R}^d) \). The **essential variation** of \( \xi \) is

\[
\text{ess}V_0^\infty(\xi) := \sup \left\{ \sum_{i=1}^n |\xi(t_i) - \xi(t_{i-1})| : t_0, \ldots, t_n \in L_\xi, \ 0 < t_0 < \cdots < t_n < \infty \right\}.
\]

An easy observation is

\[
\text{ess}V_0^\infty(\xi) \leq V_0^\infty(\xi).
\]

Combining this observation with (3.1) yields the average variation estimate

\[
\int_{\Gamma} \text{ess}V_0^\infty(\dot{\gamma}) \, d\eta(\gamma) \leq 2 \sup_{x, y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)|. \tag{4.5}
\]

Note that this estimate implies that

\[
\eta(\Gamma \setminus X) = 0
\]

where

\[
X := \{ \gamma \in \Gamma : \text{ess}V_0^\infty(\dot{\gamma}) < \infty \}.
\]

That is, \( \eta \) is concentrated on paths \( \gamma \) whose derivatives have finite essential variation.

The main reason we will work with the essential variation (rather than the usual variation) is that it has a nice lower-semicontinuity property. This follows from the identity

\[
\text{ess}V_0^\infty(\xi) = \sup \left\{ \int_0^\infty \dot{\phi}(t) \cdot \xi(t) \, dt : \phi \in C^1_c((0, \infty); \mathbb{R}^d), \|\phi\|_\infty \leq 1 \right\},
\]

which can be proved following section 5.10 of [9]. In particular, we have

\[
\liminf_{k \to \infty} \text{ess}V_0^\infty(\xi^k) \geq \text{ess}V_0^\infty(\xi) \tag{4.6}
\]

whenever \( \xi^k \to \xi \) in \( L^1_{\text{loc}}((0, \infty); \mathbb{R}^d) \). Moreover, this lower-semicontinuity immediately gives that \( X \) defined above is a Borel measurable subset of \( \Gamma \).
4.3 A converse assertion

We will now discuss a converse to the combination of Propositions 4.1 and 4.2 and its implications, which will be used in our proof of Theorem 1.4 in the next section.

**Proposition 4.7.** Assume that $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable with

$$\int_{\mathbb{R}^d} |v_0| d\rho_0 < \infty.$$ 

Further suppose that $\theta \in \mathcal{P}(\Gamma)$ satisfies the following.

(i) $e_0 \# \theta = \rho_0$.

(ii) For each $T > 0$,

$$\int_{\Gamma} \left\{ \int_0^T |\dot{\gamma}(t)|^2 dt \right\} d\theta(\gamma) < \infty.$$ 

(iii) For each $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d)$,

$$\int_0^\infty \int_{\Gamma} (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \dot{\gamma}(t) \cdot \dot{\gamma}(t)) d\theta(\gamma) dt + \int_{\Gamma} \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\theta(\gamma) = 0.$$ 

(iv) For each $t \geq 0$, set

$$\mathcal{F}(t) := \{ \gamma \in \text{supp}(\theta) : \gamma(t) \in A \} \mid A \subset \mathbb{R}^d \text{ Borel} \}.$$ 

Then $\mathcal{F}(t) \subset \mathcal{F}(s)$ whenever $0 \leq s \leq t < \infty$.

Then there is a Borel mapping $w : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ such that for each $\gamma \in \text{supp}(\theta)$

$$\dot{\gamma}(t) = w(\gamma(t), t) \quad \text{a.e. } t > 0.$$ 

Moreover, $\varrho : t \mapsto e_t \# \theta$ and $w$ is a weak solution of SPS that satisfies the initial conditions $\varrho|_{t=0} = \rho_0$ and $w|_{t=0} = v_0$.

**Proof.** 1. First we claim that

$$\mathcal{D} := \{ (\gamma, t) \in \Gamma \times [0, \infty) : \dot{\gamma}(t) \text{ exists} \}$$ 

is Borel measurable. Clearly $\mathcal{D}$ is a subset of $\Gamma \times [0, \infty)$, which is a metric space (equipped with an appropriate product metric). Let us set

$$\mathcal{D}_{n,k} := \bigcap_{\epsilon, \delta \in (0, 1/n]} \left\{ (\gamma, t) \in \Gamma \times [0, \infty) : \left| \frac{\gamma(t + \epsilon) - \gamma(t)}{\epsilon} - \frac{\gamma(t + \delta) - \gamma(t)}{\delta} \right| \leq \frac{1}{k} \right\}.$$
Note that $\mathcal{D}_{n,k}$ is an intersection of closed sets and thus itself must be closed. We leave it as an exercise to check that

$$\mathcal{D} = \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \mathcal{D}_{n,k},$$

which of course verifies measurability.

We also set

$$D(\gamma, t) = \begin{cases} \dot{\gamma}(t), & (\gamma, t) \in \mathcal{D} \\ 0, & \text{otherwise.} \end{cases}$$

Note $D(\gamma, t) = \lim_{n \to \infty} D_n(\gamma, t)$ for each $(\gamma, t) \in \Gamma \times (0, T)$, where

$$D_n(\gamma, t) = \begin{cases} n(\gamma(t + 1/n) - \gamma(t)), & (\gamma, t) \in \mathcal{D} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\mathcal{D}$ is Borel measurable.

2. Next we claim that $G := \bigcup_{\gamma \in \text{supp}(\theta)} \{(\gamma(t), t) \in \mathbb{R}^d \times [0, \infty) : t \geq 0\}$ (4.7)

is closed. To see this, suppose $(\gamma_k(t_k), t_k) \in G$ converges to $(x, t)$ as $k \to \infty$. Note that $(\gamma_k)_{k \in \mathbb{N}}$ is a uniformly Lipschitz continuous sequence

$$|\gamma_k(t_1) - \gamma_k(t_2)| \leq \left( \sup_{\text{supp}(\rho_0)} |v_0| \right) |t_1 - t_2|, \quad t_1, t_2 \geq 0.$$

Since $(\gamma_k(t_k))_{k \in \mathbb{N}}$ is convergent, it also follows that $(\gamma_k)_{k \in \mathbb{N}}$ is locally uniformly bounded. By the Arzelà-Ascoli theorem, there is a subsequence $(\gamma_{k_j})_{j \in \mathbb{N}}$ converging locally uniformly to some $\gamma \in \text{supp}(\theta)$. Then

$$x = \lim_{j \to \infty} \gamma_{k_j}(t_j) = \gamma(t).$$

Thus, $(x, t) = (\gamma(t), t) \in G$.

3. For each $n \in \mathbb{N}$, select Borel measurable maps $f_{t+1/n,t} : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$e_{t+1/n} = f_{t+1/n,t} \circ e_t$$

on $\text{supp}(\theta)$. Such maps can be shown to exist by using property (iv) as we did in Corollary (4.4). We set

$$w_n(x, t) = \begin{cases} n(f_{t+1/n,t}(x) - x), & (x, t) \in G \\ 0, & \text{otherwise} \end{cases}$$

and claim that $w_n : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ is Borel measurable. As $G$ is closed and $w_n|G^c \equiv 0$, it suffices to show that $w_n|G$ is continuous.
Select \((\gamma_k(t), t_k) \in G\) converging to \((\gamma(t), t)\) as \(k \to \infty\). As above, we can argue that there is a subsequence \((\gamma_{k_j})_{j \in \mathbb{N}}\) converging locally uniformly to some \(\gamma_\infty\) satisfying \(\gamma_\infty(t) = \gamma(t)\). By property \(iv\),

\[
\gamma_\infty(t + 1/n) = f_{t + 1/n, t}(\gamma_\infty(t)) = f_{t + 1/n, t}(\gamma(t)) = \gamma(t + 1/n).
\]

Therefore,

\[
\lim_{j \to \infty} w_n(\gamma_{kj}(t_{kj}), t_{kj}) = \lim_{j \to \infty} n(\gamma_{kj}(t_{kj} + 1/n) - \gamma_{kj}(t_{kj})) = n(\gamma_\infty(t + 1/n) - \gamma_\infty(t)) = n(\gamma(t + 1/n) - \gamma(t)) = w_n(\gamma(t), t).
\]

Since every sequence \((\gamma_k(t_k), t_k) \in G\) converging to \((\gamma(t), t) \in G\) has a subsequence \((\gamma_{kj}(t_{kj}), t_{kj})\) for which \(w_n(\gamma_{kj}(t_{kj}), t_{kj})\) converging to \(w_n(\gamma(t), t)\), it must be that \(w_n|_G\) is continuous. Consequently, \(w_n\) is Borel.

4. For every \((\gamma, t) \in D \cap (\text{supp}(\theta) \times [0, \infty))\),

\[
D(\gamma, t) = \dot{\gamma}(t)
\]

\[
= \lim_{n \to \infty} n(\gamma(t + 1/n) - \gamma(t)) = \lim_{n \to \infty} n(f_{t + 1/n, t}(\gamma(t)) - \gamma(t)) = \lim_{n \to \infty} w_n(\gamma(t), t) = \lim_{n \to \infty} w_n \circ E(\gamma, t).
\]

Here

\[
E : \Gamma \times [0, \infty) \to \mathbb{R}^d \times [0, \infty); (\gamma, t) \mapsto (\gamma(t), t)
\]

is continuous, and so

\[
G := \{(\gamma, t) \in \Gamma \times [0, \infty) : E(\gamma, t) \in B\} : \text{Borel } B \subset \mathbb{R}^d \times [0, \infty)\}
\]

is a Borel sub-sigma-algebra of the Borel subsets of \(\Gamma \times [0, \infty)\). In particular, any \(G\) measurable function is of the form \(g \circ E\) for some Borel \(g : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}\). Since \(D\) restricted to \(D \cap (\text{supp}(\theta) \times [0, \infty))\) is the pointwise limit of \(G\) measurable mappings,

\[
D|_{D \cap (\text{supp}(\theta) \times [0, \infty))} = w \circ E
\]

for some Borel measurable \(w : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^d\) (Proposition 2.7 of [10]). In particular, for every \(\gamma \in \text{supp}(\theta)\)

\[
\dot{\gamma}(t) = w(\gamma(t), t) \text{ a.e. } t > 0.
\]

3. Let us now verify \(\varrho : [0, \infty) \to \mathcal{P}(\mathbb{R}^d); t \mapsto e_{t\# \theta}\) and \(w\) is a weak solution as asserted. First, we note that since \(e_t\) is continuous, \(\varrho\) is narrowly continuous. Next by \(ii\) and Tonelli’s theorem

\[
\int_0^T \int_{\mathbb{R}^d} |w(x, t)|^2 d\varrho_t(x) dt = \int_0^T \int_{\Gamma} |w(\gamma(t), t)|^2 d\theta(\gamma) dt
\]

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\[
\begin{align*}
&= \int_0^T \int_\Gamma |\dot{\gamma}(t)|^2 d\theta(\gamma) dt \\
&= \int_\Gamma \left\{ \int_0^T |\dot{\gamma}(t)|^2 dt \right\} d\theta(\gamma) \\
&< \infty
\end{align*}
\]
for each \( T > 0 \).

Now let \( \psi \in C^\infty_c(\mathbb{R}^d \times [0, \infty)) \) and note by hypothesis (i)

\[
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \psi + \nabla \psi \cdot w) d\rho_0 dt &= \int_0^\infty \int_\Gamma (\partial_t \psi(\gamma(t), t) + \nabla \psi(\gamma(t), t) \cdot w(\gamma(t), t)) d\theta(\gamma) dt \\
&= \int_0^\infty \int_\Gamma (\partial_t \psi(\gamma(t), t) + \nabla \psi(\gamma(t), t) \cdot \dot{\gamma}(t)) d\theta(\gamma) dt \\
&= \int_\Gamma \int_0^\infty \frac{d}{dt} \psi(\gamma(t), t) dt d\theta(\gamma) \\
&= -\int_\Gamma \psi(\gamma(0), 0) d\theta(\gamma) \\
&= -\int_{\mathbb{R}^d} \psi(\cdot, 0) d\rho_0.
\end{align*}
\]

By hypothesis (iii), we have

\[
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi \cdot w + \nabla \varphi w \cdot w) d\rho_0 dt &= \int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot w(\gamma(t), t) + \nabla \varphi(\gamma(t), t) w(\gamma(t), t) \cdot w(\gamma(t), t)) d\theta(\gamma) dt \\
&= \int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \dot{\gamma}(t) \cdot \dot{\gamma}(t)) d\theta(\gamma) dt \\
&= -\int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\theta(\gamma) \\
&= -\int_{\mathbb{R}^d} \varphi(\cdot, 0) \cdot v_0 d\rho_0
\end{align*}
\]
for each \( \varphi \in C^\infty_c(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d) \).

5 Existence in one spatial dimension

Throughout this section, we will assume that \( d = 1, \rho_0 \in \mathcal{P}(\mathbb{R}) \) and \( v_0 : \text{supp}(\rho_0) \to \mathbb{R} \) is continuous and bounded. Our approach to proving Theorem 1.4 is as follows. We first select a sequence of measures \( (\rho_k^0)_{k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}) \) such that

- \( \text{supp}(\rho_k^0) \subset \text{supp}(\rho_0) \),
• $\rho^k_0$ is a convex combination of Dirac measures,
• and $\rho^k_0 \to \rho_0$ narrowly as $k \to \infty$.

It follows from the Hahn-Banach Theorem that such a sequence $(\rho^k_0)_{k \in \mathbb{N}}$ exists; for example, one may adapt Remark 5.1.2 of [1].

As explained in subsection 2.3, $(\rho^k_0)_{k \in \mathbb{N}}$ gives rise to weak solutions $\rho^k$ and $v^k$ of SPS that satisfy the initial conditions $\rho^k|_{t=0} = \rho^k_0$ and $v^k|_{t=0} = v_0$. These weak solutions in turn give rise to a sequence of probability measures $(\eta^k)_{k \in \mathbb{N}} \subset \mathcal{P}(\Gamma)$ as described in section 4. We will argue that this sequence has a subsequence that converges narrowly to some $\eta \in \mathcal{P}(\Gamma)$ that fulfills the hypotheses of Proposition 4.7. We will then conclude that $\eta$ corresponds to a weak solution $\rho$ and $v$ that satisfies the given initial conditions $\rho|_{t=0} = \rho_0$ and $v|_{t=0} = v_0$. Finally, we will verify the three estimates asserted in the statement of Theorem 1.4 for $\rho$ and $v$.

First we will need a technical lemma. Recall that since $(\rho^k_0)_{k \in \mathbb{N}}$ converges narrowly there exists a function $\Psi_0 : \mathbb{R} \to [0, \infty]$ with compact sublevel sets such that

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{R}} \Psi_0 d\rho^k_0 < \infty$$

(Remark 5.1.5 of [1]).

**Lemma 5.1.** For each $\gamma \in \Gamma$, define

$$\Psi(\gamma) := \text{ess} V^\infty_0 (\dot{\gamma}) + \|\dot{\gamma}\|_{L^\infty[0,\infty)} + \Psi_0(\gamma(0)).$$

Then $\Psi$ has compact sublevel sets in $\Gamma$ and

$$\sup_{k \in \mathbb{N}} \int_\Gamma \Psi(\gamma) d\eta^k(\gamma) < \infty.$$

**Proof.** Suppose $(\gamma^k)_{k \in \mathbb{N}} \subset \Gamma$ satisfies

$$\Psi(\gamma^k) \leq C, \quad k \in \mathbb{N}$$

for some $C \geq 0$. Then we have

$$\begin{cases} \Psi_0(\gamma^k(0)) \leq C \\ |\gamma^k(t) - \gamma^k(s)| \leq C|t - s|, \quad t, s \geq 0. \end{cases}$$

Applying a standard variant of the Arzelá-Ascoli compactness theorem, we find a subsequence $(\gamma^{kj})_{j \in \mathbb{N}}$ that converges locally uniformly to a Lipschitz continuous $\gamma : [0, \infty) \to \mathbb{R}$.

We also have that

$$\text{ess} V^\infty_0 (\dot{\gamma}^{kj}) + \|\dot{\gamma}^{kj}\|_{L^\infty[0,\infty)} \leq C.$$

Exploiting the local compactness of a sequence of functions whose essential variations are uniformly bounded (as in Theorem 4 of section 5.2 of [9]), we have that $(\dot{\gamma}^{kj})_{j \in \mathbb{N}} \subset L^2_{\text{loc}}(0, \infty)$.
has a subsequence (that we will not relabel) that converges to some $\xi$ in $L^2_{\text{loc}}(0, \infty)$. It is routine to verify that $\xi(t) = \dot{\gamma}(t)$ for almost every $t \geq 0$. Moreover, we can use the lower-semicontinuity property (4.6) and the lower-semicontinuity of the $L^\infty$ norm along almost everywhere convergent sequences of functions to show

$$
\Psi(\gamma) \leq C.
$$

Therefore, $\Psi$ has compact sublevel sets.

Recall that we have chosen $\rho^k_0$ to satisfy $\text{supp}(\rho^k_0) \subset \text{supp}(\rho_0)$. It then follows from Corollary 2.6 that

$$
\|\dot{\gamma}\|_{L^\infty[0, \infty)} \leq \sup_{x \in \text{supp}(\rho^k_0)} |v_0(x)| \leq \sup_{x \in \text{supp}(\rho_0)} |v_0(x)|
$$

for $\eta^k$ almost every $\gamma$. Employing (4.5), we also have

$$
\int_{\Gamma} \varepsilon_0 \left( \int_{\Gamma} \left| \dot{\gamma}(t) \right|^2 dt \right) d\eta^k(\gamma) \leq 2 \sup_{x, y \in \text{supp}(\rho^k_0)} |v_0(x) - v_0(y)| \leq 4 \sup_{x \in \text{supp}(\rho_0)} |v_0(x)|.
$$

As a result,

$$
\int_{\Gamma} \Psi(\gamma) d\eta^k(\gamma) \leq 5 \sup_{x \in \text{supp}(\rho_0)} |v_0(x)| + \int_{\mathbb{R}} \Psi_0 d\rho_0^k,
$$

which is uniformly bounded in $k \in \mathbb{N}$.

**Corollary 5.2.** The sequence $(\eta^k)_{k \in \mathbb{N}} \subset \mathcal{P}(\Gamma)$ has a subsequence $(\eta^{k_j})_{j \in \mathbb{N}}$ that converges narrowly to some $\eta \in \mathcal{P}(\Gamma)$. Moreover, $\eta$ satisfies the hypotheses of Proposition 4.7 below.

**Proof.** In view of Lemma 5.1, the sequence $(\eta^k)_{k \in \mathbb{N}} \subset \mathcal{P}(\Gamma)$ is tight. Since $\Gamma$ is separable, Prokhorov’s theorem implies that there is a sequence $(\eta^{k_j})_{j \in \mathbb{N}}$ that converges narrowly to some $\eta \in \mathcal{P}(\Gamma)$. We will verify that $\eta$ satisfies hypotheses (i) – (iv) of Proposition 4.7 below.

(i) As $e_0 : \Gamma \to \mathbb{R}$ is continuous, we have

$$
e_0 \# \eta = \lim_{j \to \infty} e_0 \# \eta^{k_j} = \lim_{j \to \infty} \rho^k_0 = \rho_0.
$$

(ii) In view of (5.1),

$$
\int_{\Gamma} \left\{ \int_0^T \left| \dot{\gamma}(t) \right|^2 dt \right\} d\eta^k(\gamma) = \int_0^T \int_{\Gamma} \left| \dot{\gamma}(t) \right|^2 d\eta^k(\gamma) dt
$$

$$
\leq \int_0^T \int_{\Gamma} \sup_{x \in \text{supp}(\rho_0)} |v_0|^2 d\eta^k(\gamma) dt
$$

$$
= T \sup_{x \in \text{supp}(\rho_0)} |v_0|^2.
$$
As \( \gamma \mapsto \int_0^T |\dot{\gamma}(t)|^2 dt \) is continuous and nonnegative,
\[
\int_\Gamma \left\{ \int_0^T |\dot{\gamma}(t)|^2 dt \right\} d\eta(\gamma) \leq \liminf_{j \to \infty} \int_\Gamma \left\{ \int_0^T |\dot{\gamma}(t)|^2 dt \right\} d\eta^k(\gamma) \leq T \sup_{\supp(\rho_0)} |v_0|^2 < \infty.
\]

(iii) Fix \( \varphi \in C^\infty_c(\mathbb{R}^d \times [0, \infty); \mathbb{R}^d) \). By Proposition 4.1,
\[
\int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \dot{\gamma}(t) \cdot \dot{\gamma}(t)) d\eta^k(\gamma) dt + \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\eta^k(\gamma) = 0.
\]
By our assumption that \( v_0 : \supp(\rho_0) \to \mathbb{R} \) is continuous and bounded, we have
\[
\lim_{j \to \infty} \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\eta^k(\gamma) = \lim_{j \to \infty} \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\eta(\gamma).
\]
Furthermore, the function
\[
\gamma \mapsto \int_0^\infty (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \dot{\gamma}(t) \cdot \dot{\gamma}(t)) dt
\]
is continuous, and it is not hard to check that its absolute value is also uniformly integrable with respect to the sequence \( (\eta^k)_{j \in \mathbb{N}} \). Therefore, we can send \( j \to \infty \) in (5.2) to find
\[
\int_0^\infty \int_\Gamma (\partial_t \varphi(\gamma(t), t) \cdot \dot{\gamma}(t) + \nabla \varphi(\gamma(t), t) \dot{\gamma}(t) \cdot \dot{\gamma}(t)) d\eta(\gamma) dt + \int_\Gamma \varphi(\gamma(0), 0) \cdot v_0(\gamma(0)) d\eta(\gamma) = 0.
\]
(iv) By Corollary 4.4, for each \( k \in \mathbb{N} \) and \( 0 < s \leq t \) there are Borel functions \( f_{t,s}^k : \mathbb{R} \to \mathbb{R} \) that satisfy
\[
e_t = f_{t,s}^k \circ e_s \quad (5.3)
\]
on \( \supp(\eta^k) \). In view of the quantitative sticky particle property (1.4),
\[
|f_{t,s}^k(\gamma(s)) - f_{t,s}^k(\zeta(s))| = |\gamma(t) - \zeta(t)| \leq \frac{t}{s} |\gamma(s) - \zeta(s)|
\]
for \( \gamma, \zeta \in \supp(\eta^k) \). We can also redefine \( f_{t,s}^k \) on the complement of \( e_s(\supp(\eta^k)) \) by using an appropriate Lipschitz extension of the values of \( f_{t,s}^k|_{\supp(\eta^k)} \) to obtain a globally Lipschitz continuous function which satisfies (5.3). So without any loss of generality, we may assume that
\[
|f_{t,s}^k(x) - f_{t,s}^k(y)| \leq \frac{t}{s} |x - y|
\]
for all \( x, y \in \mathbb{R} \).

By Kuratowski convergence (Proposition 5.1.8 in [1]), every \( \gamma \in \supp(\eta) \) is a limit of a sequence \( (\gamma^k_j)_{j \in \mathbb{N}} \) in \( \Gamma \) (perhaps after passing to subsequence). So
\[
|f_{t,s}^k(\gamma^k_j(s)) - \gamma^k_j(s)| = |\gamma^k_j(t) - \gamma^k_j(s)| \leq \sup_{\supp(\rho_0)} |v_0|(t - s).
\]

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It follows that \( f_{t,s}^{k_j} \) is locally uniformly bounded and uniformly equicontinuous. Passing to another subsequence if necessary, we have \( f_{t,s}^{k_j} \) converges locally uniformly on \( \mathbb{R} \) to a function \( f_{t,s} \) which satisfies
\[
|f_{t,s}(x) - f_{t,s}(y)| \leq \frac{t}{s}|x - y| \tag{5.4}
\]
for all \( x, y \in \mathbb{R} \). This local uniform convergence combined with Kuratowski convergence of \( \text{supp}(\eta^{k_j}) \) also implies that
\[
e_t = f_{t,s} \circ e_s \tag{5.5}
\]
on \( \text{supp}(\eta) \). In particular, the sigma-algebra generated by \( e_t|_{\text{supp}(\eta)} \) is included in the sigma-algebra generated by \( e_s|_{\text{supp}(\eta)} \) which implies property \((iv)\) of Proposition 4.7.

**Proof of Theorem 1.4.** By Corollary 5.2, there is a measure \( \eta \in \mathcal{P}(\Gamma) \) which fulfills the hypotheses of Proposition 4.7. As a result, there is a weak solution of SPS \( \rho \) and \( v \) that satisfies the initial conditions \( \rho|_{t=0} = 0 \) and \( v|_{t=0} = v_0 \). Moreover, \( \rho_t = e_{t\#}\eta \) for each \( t \geq 0 \) and for each \( \gamma \in \text{supp}(\eta) \),
\[
\dot{\gamma}(t) = v(\gamma(t), t), \quad \text{a.e. } t > 0. \tag{5.6}
\]
We now proceed to verifying properties \((i), (ii)\) and \((iii)\) of Theorem 1.4.

Proof of assertion \((i)\): In view of \((5.4)\) and \((5.5)\), the quantitative sticky particle property holds for \( \eta \). In particular, for each \( \gamma, \xi \in \text{supp}(\eta) \)
\[
t \mapsto \frac{1}{t}|\gamma(t) - \xi(t)|
\]
is nonincreasing. Differentiating the square of this function yields
\[
(\dot{\gamma}(t) - \dot{\xi}(t))(\gamma(t) - \xi(t)) \leq \frac{1}{t}(\gamma(t) - \xi(t))^2
\]
for almost every \( t > 0 \). In view of \((5.6)\),
\[
(v(\gamma(t), t) - v(\xi(t), t))(\gamma(t) - \xi(t)) \leq \frac{1}{t}(\gamma(t) - \xi(t))^2
\tag{5.7}
\]
for almost every \( t > 0 \).

For any \( t \geq 0 \), \( \rho_t = e_{t\#}\eta \) which implies \( \text{supp}(\rho_t) = e_{t}(\text{supp}(\eta)) \) (equation (5.2.6) of [1]). It can be shown that
\[
e_t(\text{supp}(\eta)) \text{ is closed,}
\]
similarly to showing that the set \( G \) defined in (4.7) is closed. We leave the details to the reader. Consequently, \( \text{supp}(\rho_t) = e_{t}(\text{supp}(\eta)) \). It follows that each \( x, y \in \text{supp}(\rho_t) \) is of the form \( x = \gamma(t) \) and \( y = \xi(t) \). Combining this fact with \((5.7)\) gives
\[
(v(x, t) - v(y, t))(x - y) \leq \frac{1}{t}(x - y)^2
\]
for almost every \( t > 0 \), as desired.
Proof of assertion \((ii)\): There is a function \(\vartheta : \mathbb{R} \times [0, \infty)\) such that \(\vartheta(\cdot, t) : \mathbb{R} \to \mathbb{R}\) is Borel measurable and
\[
\int_{\Gamma} v_0(\gamma(0)) h(\gamma(t)) d\eta(\gamma) = \int_{\Gamma} \vartheta(\gamma(t), t) h(\gamma(t)) d\eta(\gamma) = \int_{\mathbb{R}} \vartheta(x, t) h(x) d\rho_t(x). \tag{5.8}
\]
That is,
\[
\vartheta(e_t, t) = \mathbb{E}_\eta[v_0 \circ e_0 | e_t], \quad t \geq 0.
\]
Recall that
\[
\mathcal{E}(t) = \{ \{ \gamma \in \text{supp}(\eta) : \gamma(t) \in A \} : A \subset \mathbb{R}^d \text{ Borel} \}
\]
is the Borel sub-sigma-algebra generated by \(e_t|_{\text{supp}(\eta)}\). Moreover, \(\mathcal{E}(t) \subset \mathcal{E}(s)\) for each \(s \leq t\), so it follows from iterated conditioning that that
\[
\vartheta(e_t, t) = \mathbb{E}_\eta[\vartheta(e_s, s)|e_t], \quad s \leq t
\]
(Theorem 1.2, Chapter 4 of [7]).

Either arguing as in the proof of Corollary 2.6 or by applying the conditional Jensen’s inequality,
\[
\mathbb{E}_\eta[F(\vartheta(e_t, t))] \leq \mathbb{E}_\eta[F(\vartheta(e_s, s))]
\]
for each convex \(F : \mathbb{R} \to \mathbb{R}\) and \(s \leq t\). That is,
\[
\int_{\mathbb{R}} F(\vartheta(x, t)) d\rho_t(x) \leq \int_{\mathbb{R}} F(\vartheta(x, s)) d\rho_s(x). \tag{5.9}
\]
Consequently, in order to verify property \((ii)\), it suffices to show
\[
v(x, t) = \vartheta(x, t), \quad \rho_t \text{ a.e. } x \in \mathbb{R} \tag{5.10}
\]
for almost every \(t \geq 0\). We will argue that this is the case below.

Suppose that \(h \in C_b(\mathbb{R})\). By Corollary 4.5,
\[
\int_{\Gamma} \dot{\gamma}(t) h(\gamma(t)) d\eta^k(\gamma) = \int_{\Gamma} v_0(\gamma(0)) h(\gamma(t)) d\eta(\gamma)
\]
for each \(t \geq 0\) and each \(k \in \mathbb{N}\). We multiply this equation by \(f \in C_c^\infty(0, \infty)\) and integrate over \([0, \infty)\) to find
\[
\int_{\Gamma} \left\{ \int_0^\infty f(t) \dot{\gamma}(t) h(\gamma(t)) dt \right\} d\eta^k(\gamma) = \int_{\Gamma} \left\{ \int_0^\infty f(t) v_0(\gamma(0)) h(\gamma(t)) dt \right\} d\eta(\gamma).
\]
Sending \(k = k_j \to \infty\)
\[
\int_{\Gamma} \left\{ \int_0^\infty f(t) \dot{\gamma}(t) h(\gamma(t)) dt \right\} d\eta(\gamma) = \int_{\Gamma} \left\{ \int_0^\infty f(t) v_0(\gamma(0)) h(\gamma(t)) dt \right\} d\eta(\gamma).
\]
By (5.6), we then have
\[ \int_0^\infty f(t) \left\{ \int_\Gamma v(\gamma(t), t) h(\gamma(t)) \, d\eta(\gamma) \right\} \, dt = \int_0^\infty f(t) \left\{ \int_\Gamma v_0(\gamma(0)) h(\gamma(t)) \, d\eta(\gamma) \right\} \, dt. \]
In particular,
\[ \int_\Gamma v(\gamma(t), t) h(\gamma(t)) \, d\eta(\gamma) = \int_\Gamma v_0(\gamma(0)) h(\gamma(t)) \, d\eta(\gamma) \]
for almost every \( t \geq 0 \). Moreover,
\[ \int_\mathbb{R} v(x, t) h(x) \, d\rho_t(x) = \int_\Gamma v_0(\gamma(0)) h(\gamma(t)) \, d\eta(\gamma) \quad (5.11) \]
for almost every \( t \geq 0 \).

Assuming further that \( h \in C^1_c(\mathbb{R}) \), the conservation of momentum (part (iii) of Definition 1.1) implies that
\[ \frac{d}{dt} \int_\mathbb{R} v(x, t) h(x) \, d\rho_t(x) = \int_{\mathbb{R}} h'(x)v(x, t) \, d\rho_t(x) \]
holds in the sense of distributions on \( (0, \infty) \). Using this observation and the separability \( C^1_c(\mathbb{R}) \) we can verify the existence of a continuous \( \sigma : [0, \infty) \to (C^1_c(\mathbb{R}))' \) such that
\[ \sigma_t(h) = \int_\mathbb{R} v(x, t) h(x) \, d\rho_t(x), \quad \text{for all } h \in C^1_c(\mathbb{R}) \quad (5.12) \]
for almost every \( t \geq 0 \) (see for instance the proof of Lemma 8.1.2 of [1]). Here we consider \( C^1_c(\mathbb{R}) \) with the norm \( h \mapsto \max \{ \sup_{\mathbb{R}} |h|, \sup_{\mathbb{R}} |h'| \} \).

Combining (5.11) and (5.12) gives
\[ \sigma_t(h) = \int_\Gamma v_0(\gamma(0)) h(\gamma(t)) \, d\eta(\gamma) \quad (5.13) \]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( t \geq 0 \). The equalities (5.8), (5.11), and (5.13) then imply
\[ \int_{\mathbb{R}} v(x, t) h(x) \, d\rho_t(x) = \int_{\mathbb{R}} \overline{v}(x, t) h(x) \, d\rho_t(x) \]
for all \( h \in C^1_c(\mathbb{R}) \) and almost every \( t \geq 0 \). In particular, this equality actually holds for every bounded Borel \( h \) and almost every \( t \geq 0 \) so we conclude (5.10).

Proof of assertion (iii): 1. Set
\[ (v\#\rho)_t := \overline{v}(\cdot, t)\#\rho_t, \quad t \geq 0. \]
That is, for each \( h \in C_b(\mathbb{R}) \) and \( t \geq 0 \)
\[ \int_{\mathbb{R}} h(y) d((v\#\rho)_t)(y) := \int_{\mathbb{R}} h(\overline{v}(x, t)) d\rho_t(x). \]
By (5.9),
\[
\int_{\mathbb{R}} |y| \, d(v_{\#}\rho)_t(y) = \int_{\mathbb{R}} |\bar{v}(x,t)| \, d\rho_t(x) \leq \int_{\mathbb{R}} |v_0(x)| \, d\rho_0(x) \leq \sup_{\text{supp}(\rho_0)} |v_0| < \infty.
\]
Thus \((v_{\#}\rho)_t \in \mathcal{P}_1(\mathbb{R})\) for all \(t \geq 0\). Since we have already established that \((v_{\#}\rho)_t = v(\cdot, t)_{\#}\rho_t\)
for almost every \(t \geq 0\) (in (5.10)), we are left to show that \(v_{\#}\rho : [0, \infty) \to \mathcal{P}_1(\mathbb{R})\) is right continuous and has finite variation as defined in (1.6).

2. In order to prove that \(v_{\#}\rho : [0, \infty) \to \mathcal{P}_1(\mathbb{R})\) is right continuous, it suffices to show
\[
\lim_{t \to s^+} \int_{\mathbb{R}} F(\bar{v}(x,t)) \, d\rho_t(x) = \int_{\mathbb{R}} F(\bar{v}(x,s)) \, d\rho_s(x).
\]
(5.14)
for every \(F : \mathbb{R} \to \mathbb{R}\) that is Lipschitz continuous. In this case, we would be able to conclude that \(v_{\#}\rho\) is right continuous with respect to the narrow topology and also that
\[
\lim_{t \to s^+} \int_{\mathbb{R}} |\bar{v}(x,t)| \, d\rho_t(x) = \int_{\mathbb{R}} |\bar{v}(x,s)| \, d\rho_s(x).
\]
It would then follow that
\[
\lim_{t \to s^+} W_1((v_{\#}\rho)_t, (v_{\#}\rho)_s) = \lim_{t \to s^+} W_1(\bar{v}(\cdot, t)_{\#}\rho_t, \bar{v}(\cdot, s)_{\#}\rho_s) = 0
\]
(Proposition 7.1.5 in [1]).

By (5.8),
\[
\lim_{t \to s^+} \int_{\mathbb{R}} h(x) \bar{v}(x,t) \, d\rho_t(x) = \int_{\mathbb{R}} h(x) \bar{v}(x,s) \, d\rho_s(x)
\]
for each \(h \in C_b(\mathbb{R})\). It follows from this weak convergence that
\[
\liminf_{t \to s} \int_{\mathbb{R}} F(\bar{v}(x,t)) \, d\rho_t(x) \geq \int_{\mathbb{R}} F(\bar{v}(x,s)) \, d\rho_s(x)
\]
for each \(F : \mathbb{R} \to \mathbb{R}\) convex (Theorem 5.4.4. of [1]). In view of (5.9), we also have
\[
\limsup_{t \to s^+} \int_{\mathbb{R}} F(\bar{v}(x,t)) \, d\rho_t(x) \leq \int_{\mathbb{R}} F(\bar{v}(x,s)) \, d\rho_s(x)
\]
for each \(F : \mathbb{R} \to \mathbb{R}\) convex. Consequently, (5.14) holds for all \(F\) convex.

If \(F : \mathbb{R} \to \mathbb{R}\) is semiconvex, then \(y \mapsto F(y) + \frac{C}{2}y^2\) convex for some \(C \geq 0\). Then we may subtract the respective limits obtained in (5.14) for \(y \mapsto F(y) + \frac{C}{2}y^2\) and for \(y \mapsto \frac{C}{2}y^2\) to find that (5.14) holds for \(F\). Consequently, (5.14) holds for all \(F\) semiconvex and therefore for all \(F\) semiconcave. For \(F : \mathbb{R} \to \mathbb{R}\) Lipschitz continuous, we set
\[
F^\epsilon(y) := \inf_{z \in \mathbb{R}} \left\{ F(z) + \frac{1}{2\epsilon}(y - z)^2 \right\}.
\]

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As $F'$ is semiconcave and $F' \leq F$,
\[
\liminf_{t \to s^+} \int_{\mathbb{R}} F'(\nu(x,t))d\rho_t(x) \geq \liminf_{t \to s^+} \int_{\mathbb{R}} F'(\nu(x,t))d\rho_t(x) = \int_{\mathbb{R}} F'(\nu(x,s))d\rho_s(x).
\]
We can also use the Lipschitz assumption to verify that $\lim_{\epsilon \to 0^+} F' = F$ locally uniformly on $\mathbb{R}$. Since, $|v(x,s)| \leq \sup\{|v_0(x)| : x \in \text{supp}(\rho_0)\}$ for $\rho_s$ almost every $x \in \mathbb{R}$, we can send $\epsilon \to 0^+$ above to conclude
\[
\liminf_{t \to s^+} \int_{\mathbb{R}} F(\nu(x,t))d\rho_t(x) \geq \int_{\mathbb{R}} F(\nu(x,s))d\rho_s(x).
\]
Applying the same argument to $-F$ gives (5.14) for all Lipschitz $F$. We conclude that $v_\#\rho$ is right continuous as asserted.

3. Our final task is to verify that $v_\#\rho$ has finite variation. First recall that we have by (4.5) and the lower-semicontinuity of $\xi \mapsto \text{ess}V^\infty_0(\dot{\gamma})$
\[
\int_\Gamma \text{ess}V_0^\infty(\dot{\gamma})d\eta(\gamma) \leq \liminf_{j \to \infty} \int_\Gamma \text{ess}V_0^\infty(\dot{\gamma})d\eta_k(\gamma) \leq 2 \sup_{x,y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)|.
\]
We also have that there is a set $S \subset (0, \infty)$ of full Lebesgue measure such that for every $t \in S$:
\[
\dot{\gamma}(t) = v(\gamma(t),t) \quad \text{and} \quad \lim_{\tau \to 0^+} \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} |\dot{\gamma}(s) - \dot{\gamma}(t)|ds = 0
\]
for $\eta$ almost every $\gamma$ and
\[
v(x,t) = \nu(x,t)
\]
for $\rho_t$ almost every $x \in \mathbb{R}$.

For any $0 < t_0 < \cdots < t_n$ such that $t_i \in S$,
\[
2 \sup_{x,y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)| \geq \int_\Gamma \text{ess}V_0^\infty(\dot{\gamma})d\eta(\gamma)
\]
\[
\geq \int_\Gamma \sum_{i=1}^n |\dot{\gamma}(t_i) - \dot{\gamma}(t_{i-1})|d\eta(\gamma)
\]
\[
= \sum_{i=1}^n \int_\Gamma |\dot{\gamma}(t_i) - \dot{\gamma}(t_{i-1})|d\eta(\gamma)
\]
\[
= \sum_{i=1}^n \int_\Gamma |v(\gamma(t_i), t_i) - v(\gamma(t_{i-1}), t_{i-1})|d\eta(\gamma)
\]
\[
= \sum_{i=1}^n \int_{\mathbb{R} \times \mathbb{R}} |z_i - z_{i-1}|d\pi^i(z_i, z_{i-1})
\]
Here
\[
\pi^i := (v(e_{t_i}, t_i), v(e_{t_{i-1}}, t_{i-1})) \# \eta
\]
is a Borel probability measure on \( \mathbb{R} \times \mathbb{R} \) with first marginal
\[
v(e_{t_i}, t_i) \# \eta = v(\cdot, t_i) \# \rho_{t_i} = v(\cdot, t_i) \# \rho_{t_i}
\]
and second marginal
\[
v(e_{t_{i-1}}, t_{i-1}) \# \eta = v(\cdot, t_{i-1}) \# \rho_{t_{i-1}} = v(\cdot, t_{i-1}) \# \rho_{t_{i-1}}.
\]
Therefore,
\[
\sum_{i=1}^{n} W_1 ((v \# \rho)_{t_i}, (v \# \rho)_{t_{i-1}}) \leq \sum_{i=1}^{n} \int_{\mathbb{R} \times \mathbb{R}} |z_i - z_{i-1}| d\pi^i(z_i, z_{i-1}) \\
\leq 2 \sup_{x,y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)| . \tag{5.15}
\]

Now suppose \( 0 < t_0 < \cdots < t_n < t_{n+1} \) are arbitrary. Since \( S \) has full Lebesgue measure in \( (0, \infty) \), it is possible to select sequences \((s^k_i)_{k \in \mathbb{N}} \subset S\) that satisfy
\[
t_i < s^k_i < t_{i+1}
\]
for \( i = 0, \ldots, n \). We can then apply \( \tag{5.15} \) with \( 0 < s^k_0 < \cdots < s^k_n \) for each \( k \in \mathbb{N} \) to get
\[
\sum_{i=1}^{n} W_1 ((v \# \rho)_{s^k_i}, (v \# \rho)_{s^k_{i-1}}) \leq 2 \sup_{x,y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)| .
\]

Letting \( k \to \infty \) and using the right continuity of \( v \# \rho \) gives that \( \tag{5.15} \) holds for the arbitrarily selected \( 0 < t_0 < \cdots < t_n \). As a result,
\[
V_0^\infty (v \# \rho) \leq 2 \sup_{x,y \in \text{supp}(\rho_0)} |v_0(x) - v_0(y)| .
\]

\[\square\]

References


